Diseño estabilizante de controladores predictivos para regulación y seguimiento

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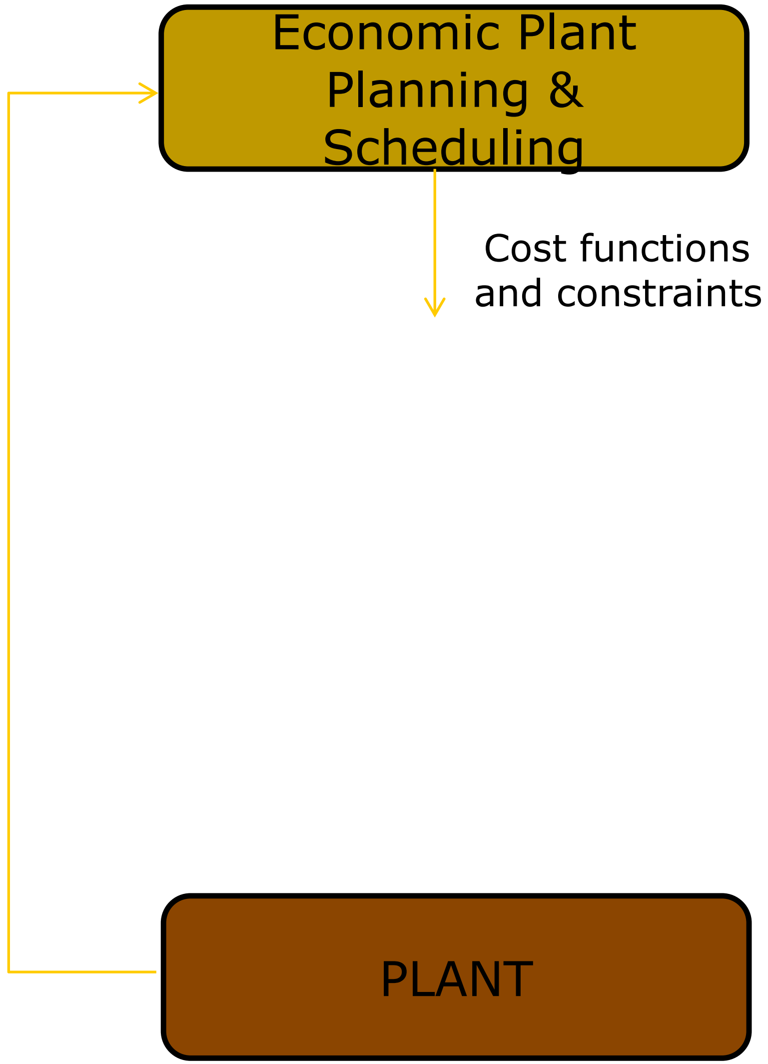
Outline

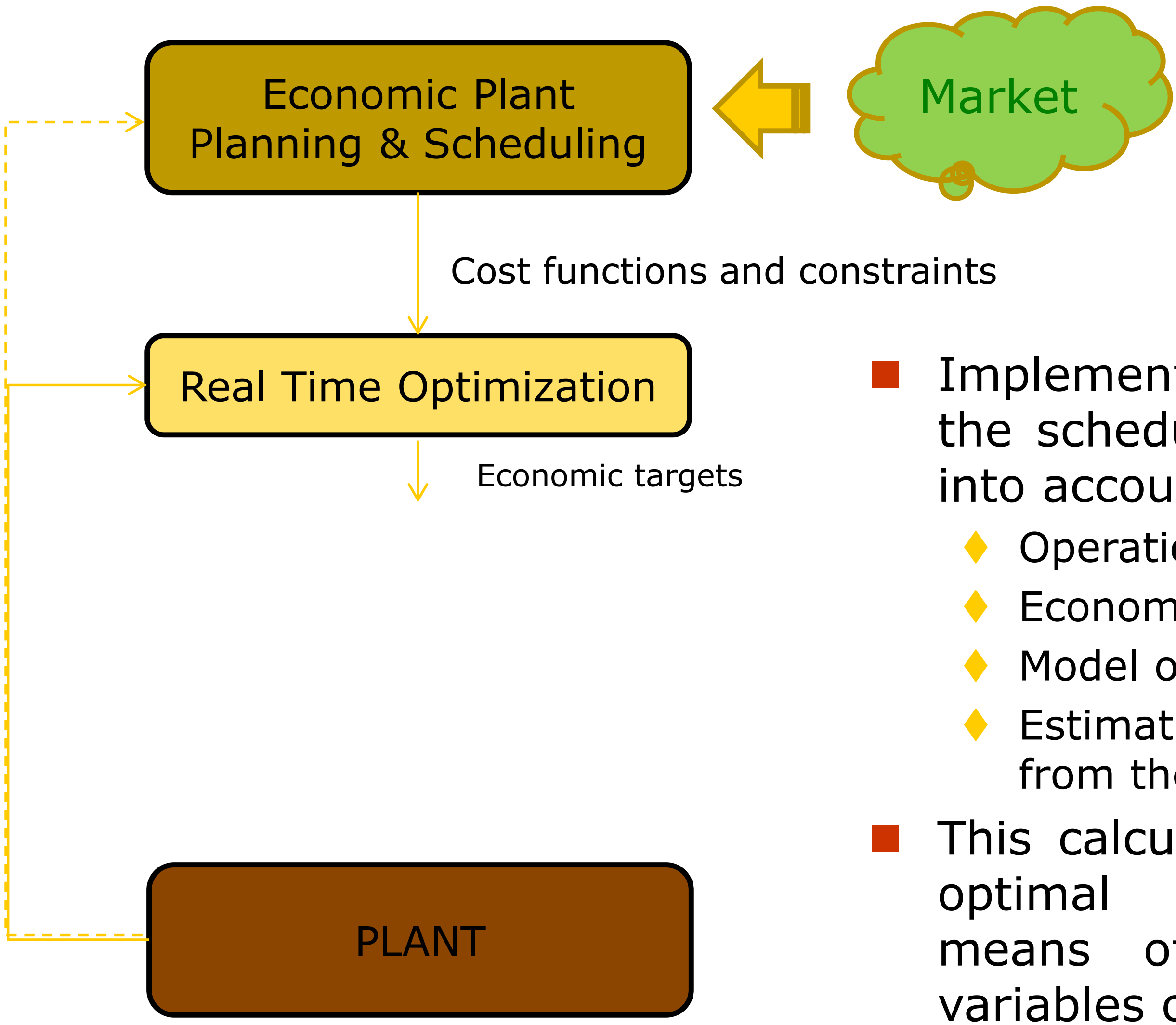
- **Stabilizing design of predictive controllers**
- Tracking model predictive control
- Economic model predictive control
- Conclusions

Optimal operation of a plant



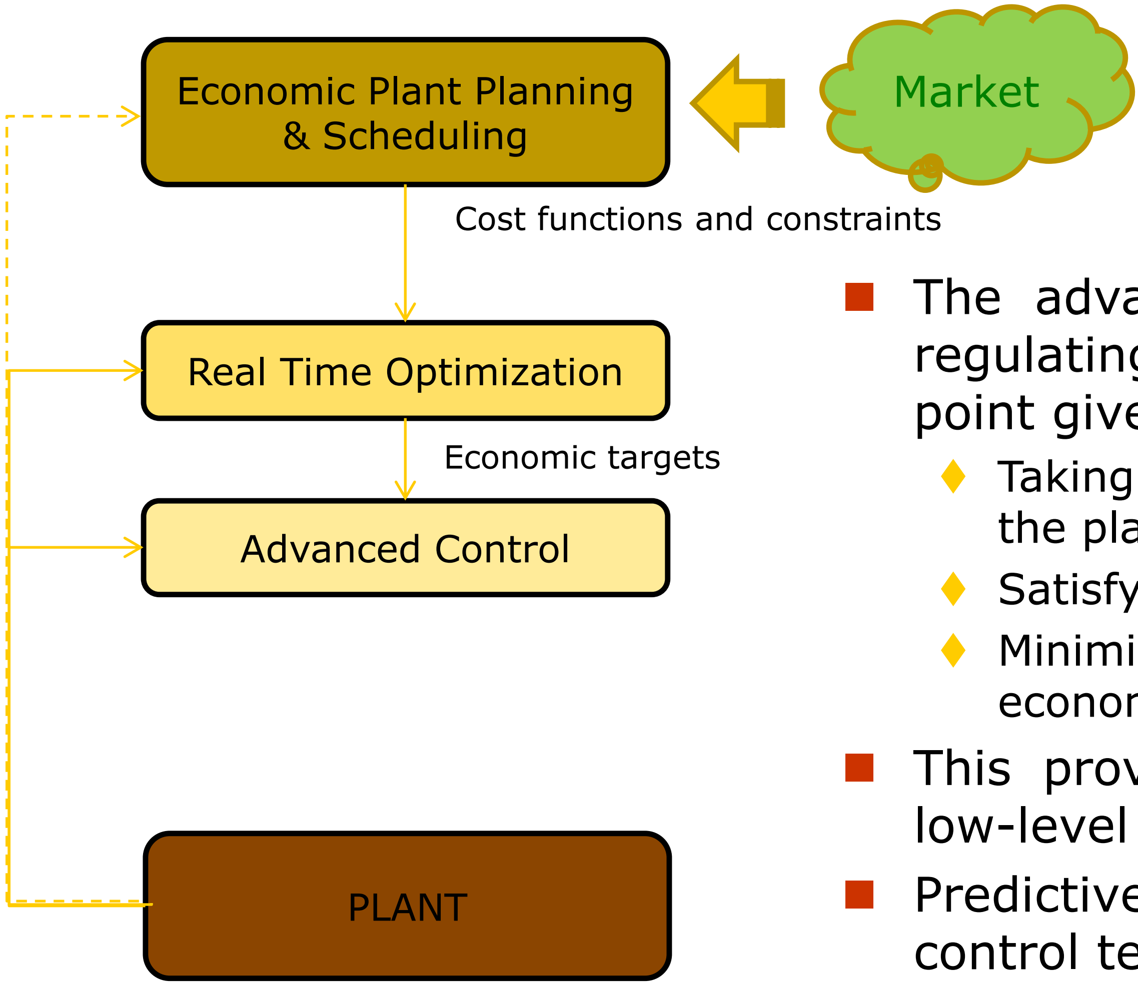
- Governed by markets, contracts with clients and suppliers
- It is calculated what, when and how much to produce
- The work plan is calculated
  - Cost function
  - Operation limits

Optimal operation of a plant



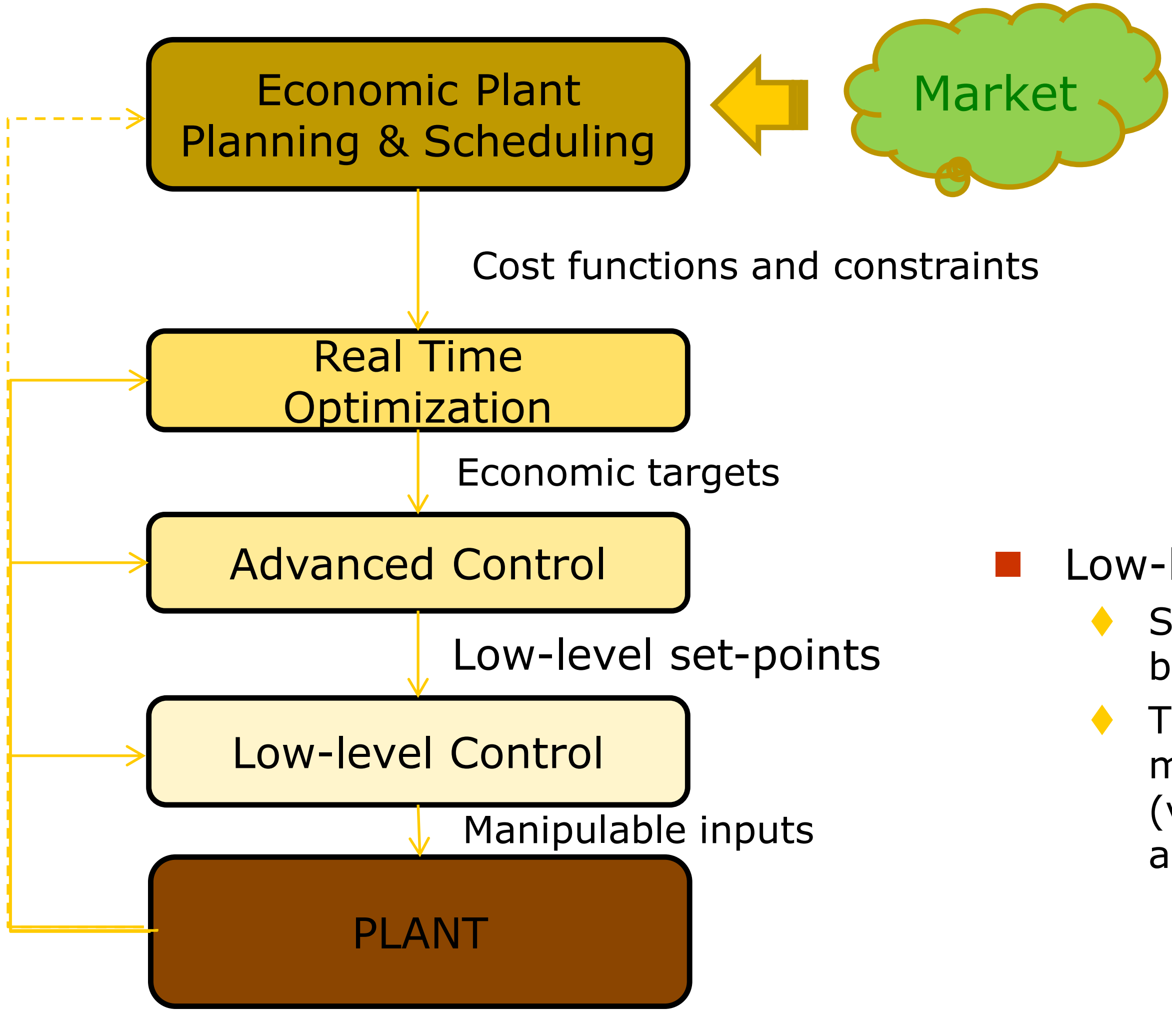
- Implementation in real time of the scheduled work plan, taking into account
  - Operation limits
  - Economic cost of the operation
  - Model of the plant
  - Estimated and reconciliated data from the plant
- This calculates the economically optimal operating point by means of targets of certain variables of the process

Optimal operation of a plant



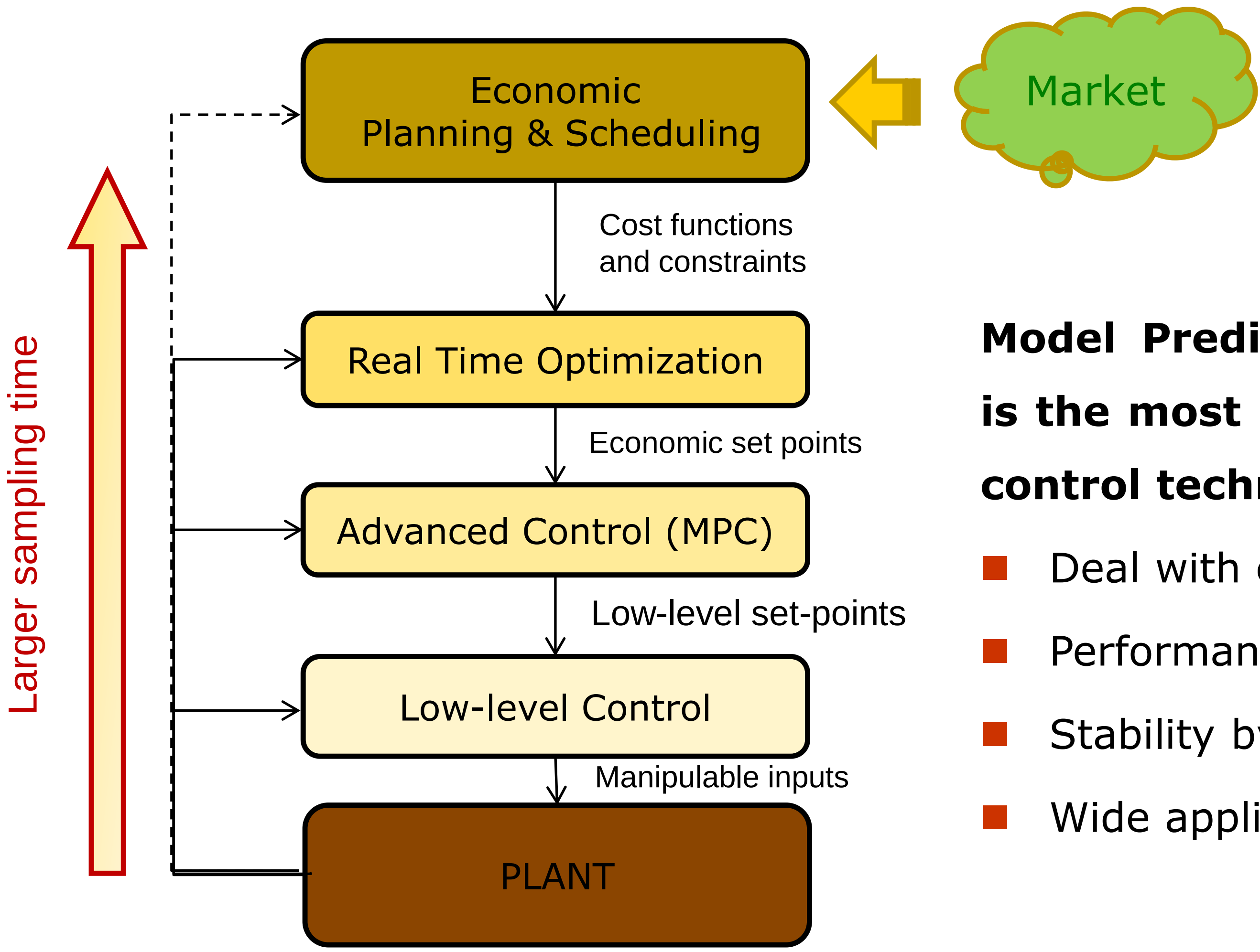
- The advanced control is aimed at regulating the plant in the operating point given by the RTO
  - Taking into account the dynamics of the plant
  - Satisfying the hard constraints
  - Minimizing the error w.r.t. the economic targets
- This provides the setpoints to the low-level control loops
- Predictive control is the most used control technique

Optimal operation of a plant



- Low-level control
  - Simple control loops based on PIDs
  - The values of the manipulable variables (valve position, etc) are applied.

Optimal operation of a plant



Model Predictive Control (MPC) is the most successful advanced control technique

- Deal with constraints
- Performance optimization
- Stability by design
- Wide application field

Problem statement

- Consider a system given by

$$x^+ = f(x,u)$$

  - The model function can describe complex dynamics, such as discontinuous dynamics, such as switching systems or hybrid dynamics

- The origin is the target equilibrium point $f(0,0) = 0$.

- The system is subject to **hard constraints** that limit the states and inputs at each sampling time

$$(x,u) \in Z$$

- The MPC is a state feedback control law $u = \kappa(x)$.

Problem statement

- Let $u = \kappa(x)$ be the MPC control law, then the closed loop system is given by

$$x^+ = f(x, \kappa(x)) = F(x)$$

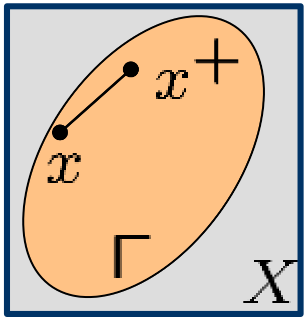
subject to the constraints

$$x \in X_\kappa = \{x : (x, \kappa(x)) \in Z\}$$

Stability conditions of constrained nonlinear systems

Lyapunov stability of constrained systems

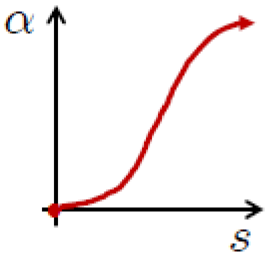
- Consider the system $x^+ = F(x)$ subject to $x \in X$
- The origin is an equilibrium point $F(0) = 0$
- A set Γ is a positively invariant (PI) set iff
  - For all $x(0) \in \Gamma$, $x(k) \in \Gamma$ for all $k \geq 0$.
  - For all $x \in \Gamma$, $x^+ = F(x) \in \Gamma$



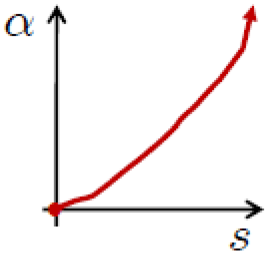
- An PI set Γ is admissible iff $\Gamma \subseteq X$
- **Constraint satisfaction $\Leftrightarrow$ Admissible PI** (well posedness)
- Assumption: Γ is closed

Comparison functions

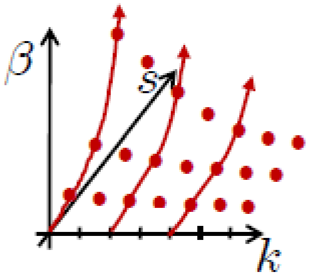
- Functions $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$
  - A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathcal{K}$ function if it is continuous, $\alpha(0) = 0$ and strictly increasing.



  - A function α is a $\mathcal{K}_\infty$ function if it is $\mathcal{K}$ and $\alpha(s) \to \infty$ when $s \to \infty$.



  - A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathcal{KL}$ function if for all $t \in \mathbb{R}^+$, $\beta(\cdot, t)$ is $\mathcal{K}$ and for all $s \in \mathbb{R}^+$ $\beta(s,t) \to 0$ when $t \to \infty$



Lyapunov function

- Let Γ be an admissible PI set. Let Ω be a set $\Omega \subseteq \Gamma$.
  Then $V(x) : R^n \to \mathbb{R}_+$ is a Uniform Strict Lyapunov Function iff

$$\begin{aligned} V(x) &\geq \alpha_1(\|x\|), \quad \forall x \in \Gamma \\ V(x) &\leq \alpha_2(\|x\|), \quad \forall x \in \Omega \\ V(F(x)) - V(x) &\leq -\alpha_3(\|x\|), \forall x \in \Gamma \end{aligned}$$

  where $\alpha_1, \alpha_2, \alpha_3$ are $\mathcal{K}$ functions.

- If a system admits a USLF then it is

  asymptotically stable (AS) in Γ

Lyapunov function

- The system is $\mathcal{KL}$ AS iff there exists a $\mathcal{KL}$ function such that

$$\|x(k)\| \leq \beta(\|x(0)\|, k), \; \forall x(0) \in \Gamma$$

- We are interested in $\mathcal{KL}$ AS since
  - Provides a bound of the state evolution
  - Allows converse Lyapunov theorems
  - Allows to derive inherent robustness (ISS)
- AS and $\mathcal{KL}$AS are not equivalent since $F(x)$ or $V(x)$ might be discontinuous .
- Stronger conditions are required for $\mathcal{KL}$AS:

  α_1 and α_3 are $\mathcal{K}_\infty$ functions and $V(x)$ is locally bounded* in Γ

(* Any compact set is mapped in a compact set)

Model predictive control

- It is a model-based optimal control law that miminizes the predicted performance.

- Ingredients:
  - Prediction model $x^+ = f(x,u), \quad (x,u) \in Z.$
  - Predictor for a given state x and a sequence of N future control inputs $\mathbf{u}$:

$$x(j) = \phi(j, x, \mathbf{u}), \quad j \in \{0, 1, \cdots, N\}$$

  - Stage cost function $L(x,u)$: measures the performance of the plant at (x,u).
  - The cost of the predicted trajectory along the prediction horizon N is minimized

$$V_N(x, \mathbf{u}) = \sum_{j=0}^{N} L(x(j), u(j))$$

The optimal controller

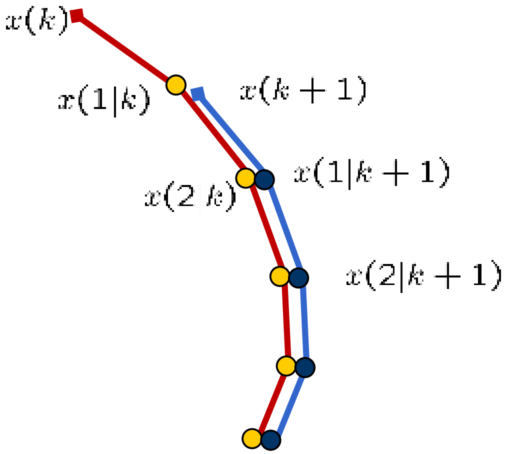
- Optimal controller: infinite prediction horizon ($N = \infty$)
- The optimal control law is derived from the solution of

$$
\begin{aligned}
\min_{\mathbf{u}} \quad & V_\infty(x, \mathbf{u}) \triangleq \sum_{j=0}^{\infty} L(x(j), u(j)) \\
s.t. \quad & x(j) = \phi(j, x, \mathbf{u}),\ j \geq 0 \\
& (x(j), u(j)) \in Z,\ j \geq 0
\end{aligned}
$$

- Calculation of the control law
  - The control law function $\kappa_\infty(x)$ from the HJB optimality conditions
  - The control action $u(x)$ numerically from the solution of a parametric optimization problem (with infinite decision variables)

The optimal controller

- Bellman's optimality conditions
  - $x(j|k) = x(j-1|k+1) = x(k+j)$
  - $V^o_\infty(x(k)) = L(x(k), u^o(k)) + V^o_\infty(x(k+1))$



- Feasibility
  - The solution satisfy that $V^o_\infty(x) < \infty$
  - If $V^o_\infty(x) < \infty$, then $x(j) \to 0$
  - Then the feasibility region X_∞ is the set of states that can be asymptotically stabilized satisfying the constraints.
- The feasibility region X_∞ is a positive invariant set of the closed-loop system

The optimal controller

- Stability
  - The system is locally stabilizable
    - $\exists u = \kappa_f(x)$ such that $x^+ = f(x, \kappa_f(x))$ is L$\mathcal{KL}$AS.
    - There exists a USLF $V_f(x)$.
    - $V_\infty^o(x) \leq V_f(x) \leq \alpha_2^f(\|x\|)$ for all $x \in \Omega$.
  - $L(x,u) \geq \alpha_1(\|x\|)$ where α_1 is a $\mathcal{K}$-function
    - $V_\infty^o(x) \geq L(x,u) \geq \alpha_1(\|x\|)$
  - $V_\infty^o(F(x)) - V_\infty^o(x) = -L(x, \kappa_\infty(x)) \leq -\alpha_1(\|x\|)$
  - $V_\infty^o(x)$ is a USLF in $X_\infty \Rightarrow$ AS in X_∞

Stabilizing nominal MPC

- The MPC is a practical approximation of the optimal controller
  - Finite prediction horizon N
  - The optimization problem $P_N(x)$ is solved numerically
  - Application using a receding horizon technique

- The MPC optimization problem $P_N(x)$ is

$$
\begin{aligned}
\min_{\mathbf{u}} \quad & V_N(x,\mathbf{u}) \triangleq \sum_{j=0}^{N-1} L(x(j),u(j)) + V_f(x(N)) \\
s.t. \quad & x(j) = \phi(j,x,\mathbf{u}),\ j = 0,\cdots,N-1 \\
& (x(j),u(j)) \in Z,\ j = 0,\cdots,N-1 \\
& x(N) \in X_f
\end{aligned}
$$

Stabilizing Terminal Ingredients

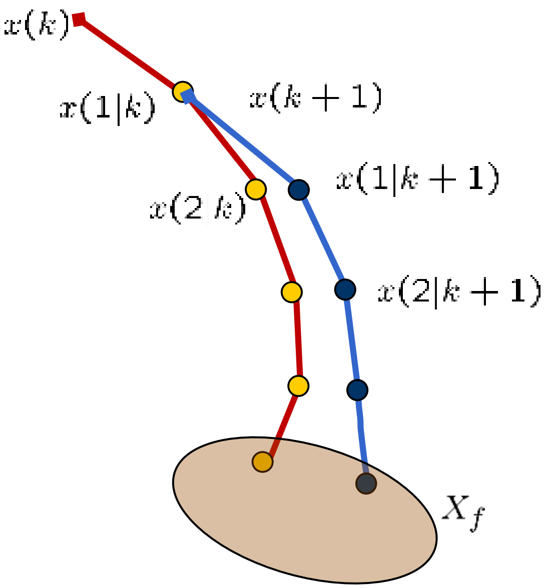
X_N is the set of states x where the $P_N(x)$ is feasible

Stabilizing nominal MPC

- Receding horizon technique: at each sampling time k
  - $P_N(x(k))$ is solved $\Rightarrow \mathbf{u}^o(x(k))$
  - Only the current input is applied

$$u(k) = u^o(0|k) = \kappa_N(x(k))$$

This provides feedback.



- Stability issues
  - $x(j|k) \neq x(j-1|k+1) \neq x(k+j)$
  - Nice properties of the optimal controller does not hold anymore

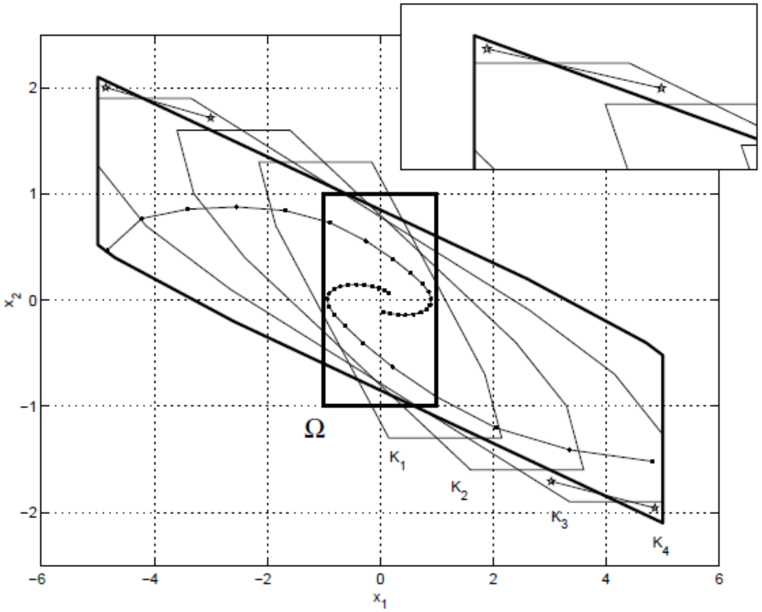
Stabilizing nominal MPC

- Recursive feasibility $\forall x \in X_N,\ f(x, \kappa_N(x)) \in X_N$:

  The feasibility region must be a PI set

- This property does not hold in general

If at sampling time k
X_f can be reached in N steps,
then at next sampling time $k+1$
X_f can be reached in $N-1$ steps
But
X_f might be not reachable in N steps



Stabilizing nominal MPC

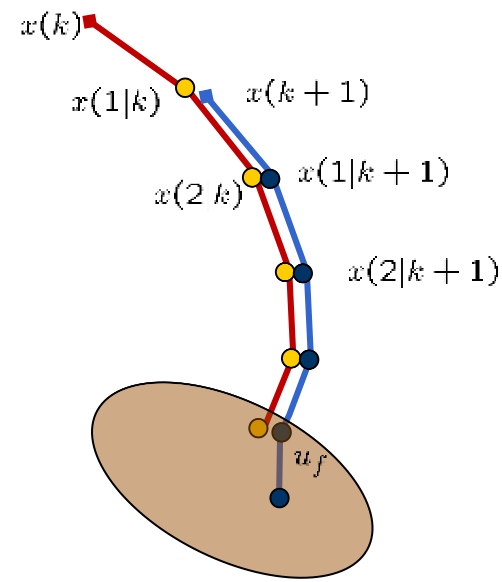
- Assume that X_f is an admissible PI set, then

  $$\forall x \in X_f, \exists u_f \text{ such that } (x, u_f) \in Z \text{ and } f(x, u_f) \in X_f$$

- For a given $x(k)$, $x(N|k) \in X_f$.
  Let u_f be such that:
  $(x(N|k), u_f) \in Z$ and $f(x(N|k), u_f) \in X_f$
  then

  $$\mathbf{u}(k+1) = \{u^o(1|k), \cdots, u^o(N-1|k), u_f\}$$

  is a feasible solution at $x(k+1)$.



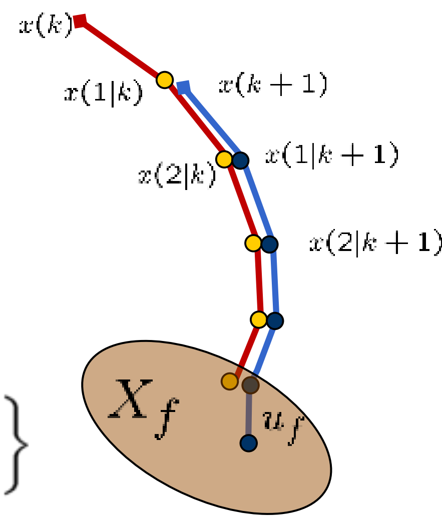
- $P_N(x(k+1))$ is feasible
  $\mathbf{u}(k+1)$ is called the shifted solution

Stabilizing nominal MPC

- Calculate $\tilde{\Delta}V = V_N(x(k+1),\mathbf{u}(k+1)) - V_N^o(x(k))$

For $j \in [1, N-1]$, $x(j-1|k+1) = x^o(j|k)$ and $u(j-1|k+1) = u^o(j|k)$

then



$$
\begin{aligned}
\tilde{\Delta}V &= \left\{\sum_{j=0}^{N-2} L(x(j|k+1),u(j|k+1))\right.\\
&\quad \left. +L(x(N-1|k+1),u_\Omega) + V_f(x(N|k+1))\right\}\\
&\quad -\left\{L(x(k,u(k)) + \sum_{j=1}^{N-1} L(x(j|k),u(j|k)) + V_f(x(N|k))\right\}\\
&= -L(x(k,u(k))\\
&\quad +\left\{L(x(N|k),u_f) + V_f(x(N|k+1)) - V_f(x(N|k))\right\}
\end{aligned}
$$

Stabilizing nominal MPC

- $V_f(x)$ must be such that for all $x \in X_f$, there exists $u = \kappa_f(x)$
  - $(x, \kappa_f(x)) \in Z,\ f(x, \kappa_f(x)) \in X_f$
  - $L(x, \kappa_f(x)) + V_f(f(x, \kappa_f(x))) - V_f(x) \leq 0$

As $x(N|k) \in X_f$, then

$$L(x(N|k), u_f) + V_f(x(N|k+1)) - V_f(x(N|k)) \leq 0$$

This implies that

$$\begin{aligned} \tilde{\Delta} V &= -L(x(k, u(k)) \\ &\quad + \underbrace{\left\{ L(x(N|k), u_f) + V_f(x(N|k+1)) - V_f(x(N|k)) \right\}}_{\leq 0} \\ &\leq -L(x(k, u(k)) \end{aligned}$$

Stabilizing nominal MPC

- We have that

$$V_N(x(k+1), \mathbf{u}(k+1)) - V_N^o(x(k)) \leq -L(x(k), u^o(k))$$

If the optimal solution at $x(k+1)$ is applied, then

$$V_N^o(x(k+1)) \leq V_N(x(k+1), \mathbf{u}(k+1))$$

and consequently,

$$V_N^o(x(k+1)) - V_N^o(x(k)) \leq -L(x(k), u^o(k))$$

Stabilizing nominal MPC

- We have that
  - X_N is a PI for the controlled system
  - $V_N^o(x) \geq L(x,u) \geq \alpha_1(\|x\|)$ for all $x \in X_N$
  - $V_N^o(x) \leq V_f(x) \leq \alpha_2(\|x\|)$ for all $x \in X_f$
  - $V_N^o(x^+) - V_N^o(x) \leq -L(x,\kappa_N(x)) \leq -\alpha_1(\|x\|)$ for all $x \in X_N$

- Then the optimal cost function $V_N^o(x)$ is a Lyapunov function and

  The controlled system is AS in X_N

Conditions for $\mathcal{KL}$AS

- The control law $\kappa_N(x)$ may be discontinuous (even when the functions of $P_N(x)$ are continuous).

- Stronger conditions for $\mathcal{KL}$AS(X_N):
  - If $\alpha_1 \in \mathcal{K}_\infty$ and $V_N^o(x)$ is Locally Bounded in X_N.
  - If $\alpha_1 \in \mathcal{K}_\infty$ and there exists M such that $V_N^o(x) \leq M$ for all $x \in X_N$,
    - If $f(x,u)$, $L(x,u)$ and $V_f(x)$ are continuous functions and Z is bounded then $V_N^o(x)$ is locally bounded in Z.
  - If $f(x,u)$ is linear, $L(x,u)$ and $V_f(x)$ convex functions and Z a convex set then $V_N^o(x)$ is locally bounded in Z.

Stability with terminal equality constraint

- A simple design of stabilizing MPC
  - $V_f(x) = 0$
  - $X_f = \{0\}$

$$\begin{aligned}
\min_{\mathbf{u}} \quad & V_N(x,\mathbf{u}) \triangleq \sum_{j=0}^{N-1} L(x(j),u(j)) \\
s.t. \quad & x(0) = x \\
& x(j+1) = f(x(j),u(j), \; j = 0,\cdots,N-1 \\
& (x(j),u(j)) \in Z, \; j = 0,\cdots,N-1 \\
& x(N) = 0
\end{aligned}$$

- Stability issue: Upper bound of $V_N^o(x)$
  - Weak controllability assumption: $\exists \alpha_2 \in \mathcal{K}$ such that $V_N^o(x) \leq \alpha_2(|x|)$ locally.
  - This condition can be added as a constraint.

Stabilizing MPC without terminal constraint

- Terminal constraint is the most difficult ingredient to calculate.
- Is the MPC stabilizing if the terminal constraint is removed?
- Idea:

$$\begin{aligned} \min_{\mathbf{u}} \quad & V_N(x,\mathbf{u}) \triangleq \sum_{j=0}^{N-1} L(x(j),u(j)) + \lambda V_f(x(N)) \\ s.t. \quad & x(0) = x \\ & x(j+1) = f(x(j),u(j)),\ j = 0,\cdots,N-1 \\ & (x(j),u(j)) \in Z,\ j = 0,\cdots,N-1 \end{aligned}$$

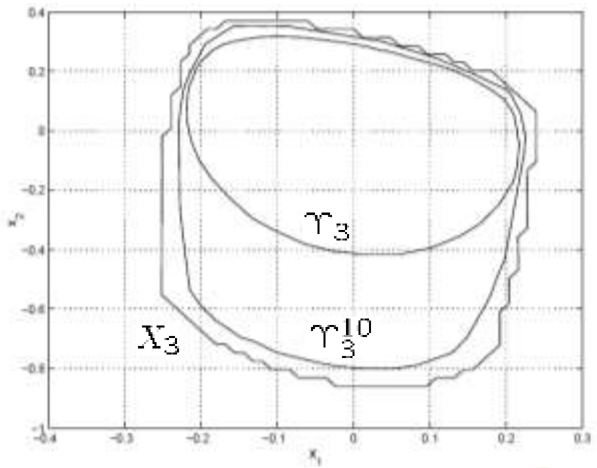
Stabilizing MPC without terminal constraint

- There exists a level set $\Upsilon_N(\lambda)$, such that for all $x \in \Upsilon_N(\lambda)$, the terminal constraint is not active (Then can be removed from $P_N(x)$)

- $\Upsilon_N(\lambda)$ is a PI set for the controlled system

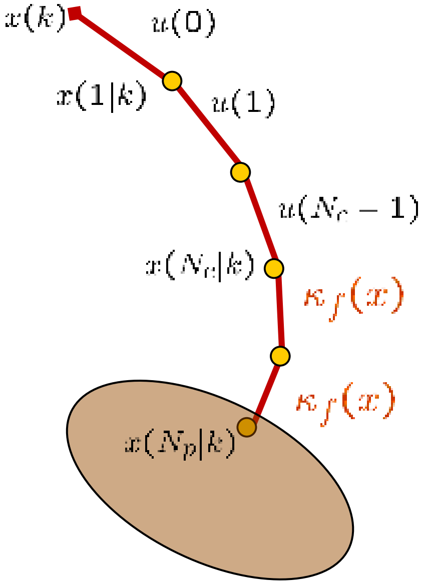
- The domain of attraction can be enlarged by weighting the cost function:

$$\Upsilon_N(\lambda_1) \subseteq \Upsilon_N(\lambda_2),\ \forall \lambda_1 \leq \lambda_2$$



Prediction and control horizon

- A prediction horizon N_p larger than the control horizon N_c
  - Larger predicted trajectory in the cost
  - Limit the number of decision variables
- Idea: Use the terminal control law to fill the gap *[Magni'01]*
- Cost function



$$V_{N_c,N_p}(x,\mathbf{u}) = \sum_{i=0}^{N_c-1} L(x(i),u(i)) + \sum_{i=N_c}^{N_p-1} L(x(i),\kappa_f(x(i))) + V_f(x(N_p))$$

Prediction and control horizon

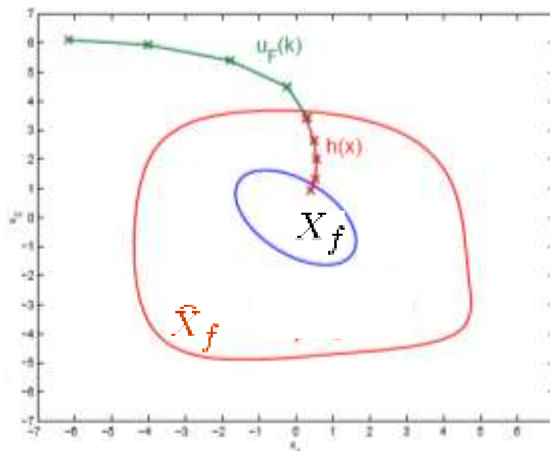
$$\min_{\mathbf{u}} \quad V_{N_c,N_p}(x,\mathbf{u}) = \sum_{i=0}^{N_c-1} L(x(i),u(i)) + \sum_{i=N_c}^{N_p-1} L(x(i),\kappa_f(x(i))) + V_f(x(N_p))$$

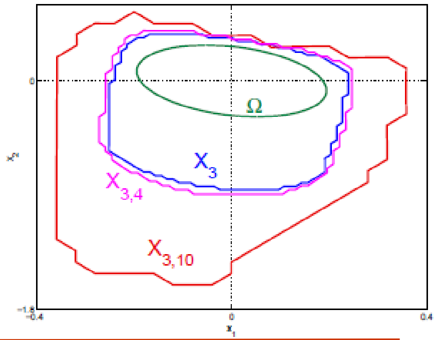
$$\hat{V}_f(x(N_c))$$

$$\begin{aligned} s.t. \quad & x(0) = x \\ & x(i+1) = f(x(i),u(i)), \quad i = 1,\cdots,N_c - 1 \\ & (x(i),u(i)) \in Z, \quad i = 1,\cdots,N_c - 1 \end{aligned}$$

$$\begin{aligned} & x(i+1) = f(x(i),\kappa_f(x(i))), \quad i = N_c,\cdots,N_p \\ & (x(i),\kappa_f(x(i))) \in Z, \quad i = N_c,\cdots,N_p - 1 \\ & x(N_p) \in X_f \end{aligned}$$

$$x(N_c) \in \hat{X}_f$$





- Advantages:
  - Larger domain of attraction $X_f \subset\subset \hat{X}_f$
  - Better closed loop performance.

Local optimality

- The optimal controller is $\kappa_\infty(x)$

  (The MPC control law $\kappa_N(x)$ is sub-optimal)

- The infinite horizon cost of the trajectory of the closed-loop system is

  $$V_\infty^{\kappa_N}(x) = \sum_{j=0}^{\infty} L(x(j), \kappa_N(x(j)))$$

  Notice that $V_\infty^{\kappa_N}(x) \neq V_N^o(x)$

- Closed-loop performance

  $$V_N^o(x) \geq V_\infty^{\kappa_N}(x) \geq V_\infty^o(x), \quad \forall x \in X_N$$

  The performance of the closed-loop system is better than predicted.

Local optimality

- How could the optimality gap be reduced?
  - Enlarging N:

$$V_N^o(x) \geq V_{N+1}^o(x) \geq V_\infty^o(x)$$

  - Taking a larger prediction horizon:

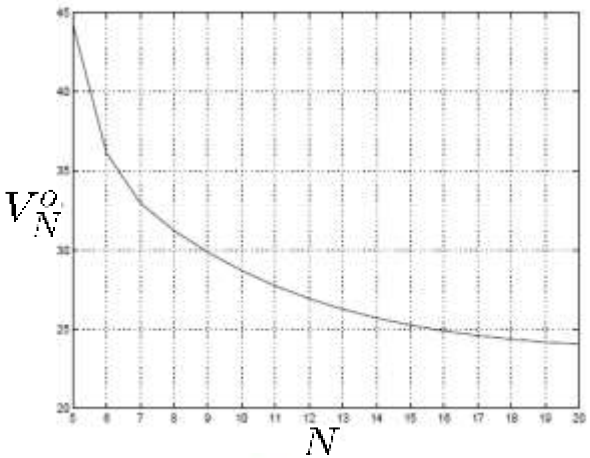
$$V_{N_c}^o(x) \geq V_{N_c,N_p}^o(x) \geq V_\infty^{\kappa_{N_c,N_p}}(x) \geq V_\infty^o(x)$$

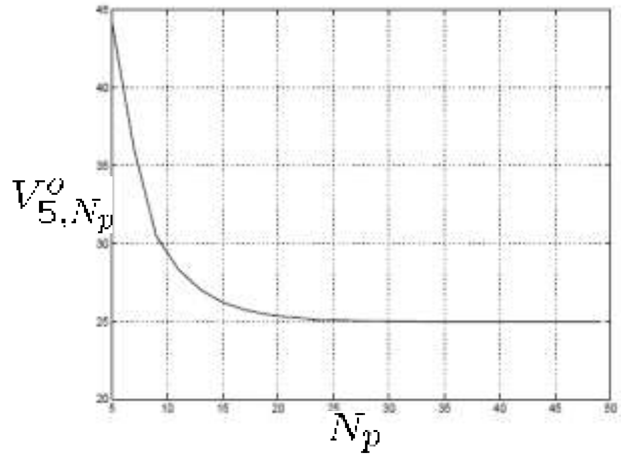
  - Taking $\kappa_f(x) = \kappa_\infty(x)$ and $V_f(x) = V_\infty^o(x)$ locally:

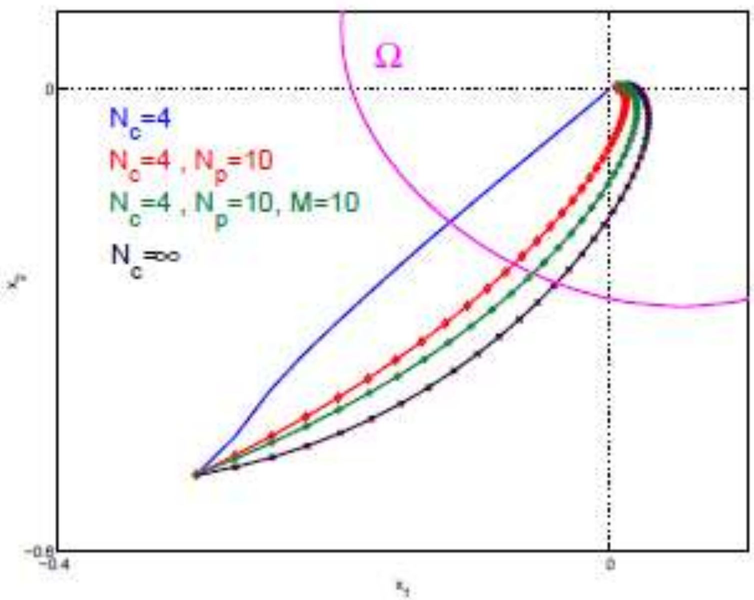
$V_N^o(x) = V_\infty^o(x)$ when the terminal constraint is not active

Local optimality property

Illustrative example: CSTR







$V_4^o(x_0)$	33.5711
$V_{4,10}^o(x_0)$	29.4109
$V_\infty^{K4,10}(x_0)$	23.6733
$V_\infty^o(x_0)$	23.5891

Outline

- Stabilizing design of predictive controllers
- **Tracking model predictive control**
- Economic model predictive control
- Conclusions

Model Predictive Control

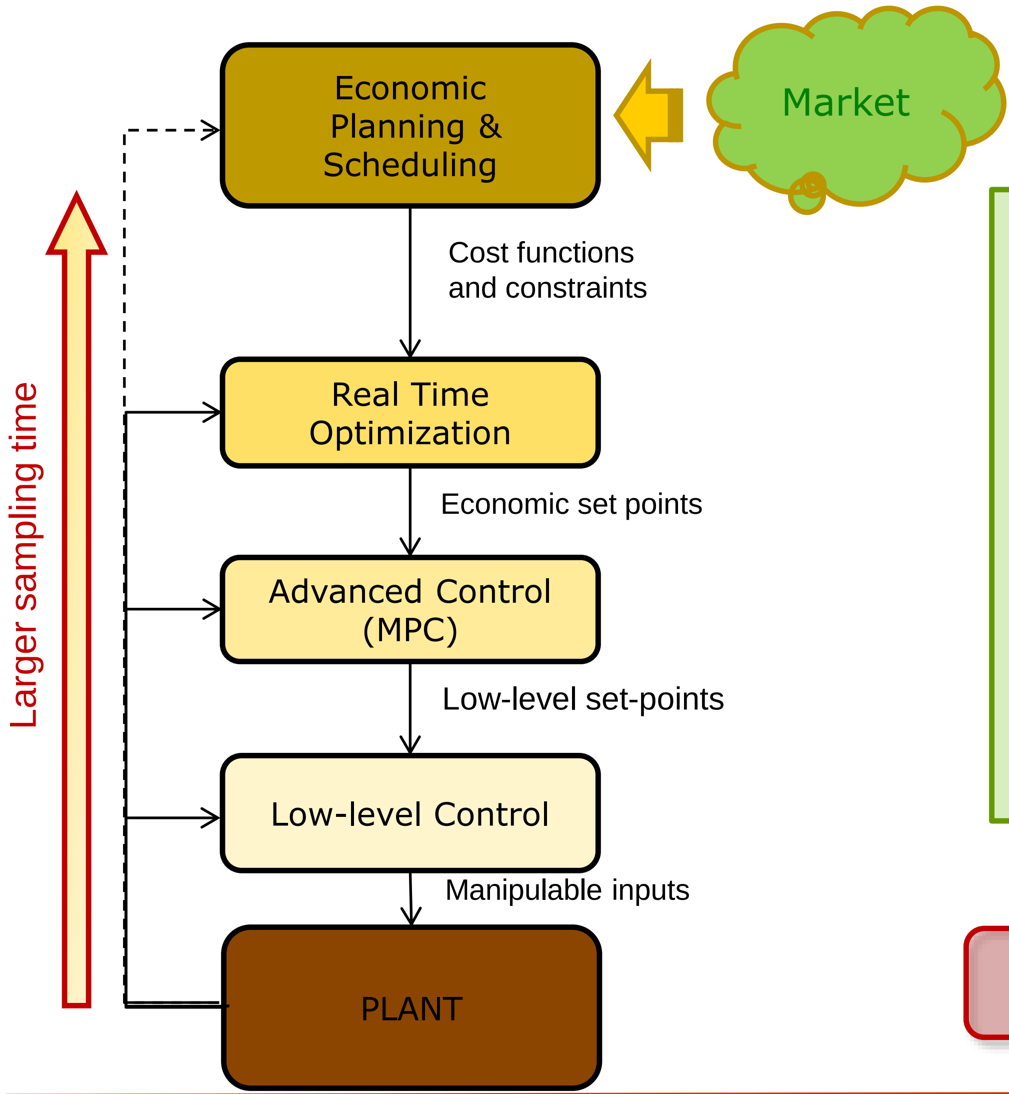
- Objective: regulate the system to the MPC target

$$
\begin{aligned}
\min_{\mathbf{u}} \quad & V_N(x,\mathbf{u}) = \sum_{i=0}^{N-1} \ell(x(i)-x_s^*, u(i)-u_s^*) + V_f(x(N)-x_s^*) \\
s.t. \quad & x(i+1) = f(x(i),u(i)) \\
& x(0) = x \\
& u(j) \in \mathcal{U},\ x(j) \in \mathcal{X}, \qquad j = 0,\cdots,N-1. \\
& x(N) - x_s^* \in \Omega
\end{aligned}
$$

  - $\ell(x-x_s^*, u-u_s^*)$ measures the traking error
- The optimal predicted sequence $\mathbf{u}^*(x)$ is computed
- Receding horizon

$$\kappa_N(x) = \mathbf{u}^*(0;x)$$

Changing set-points



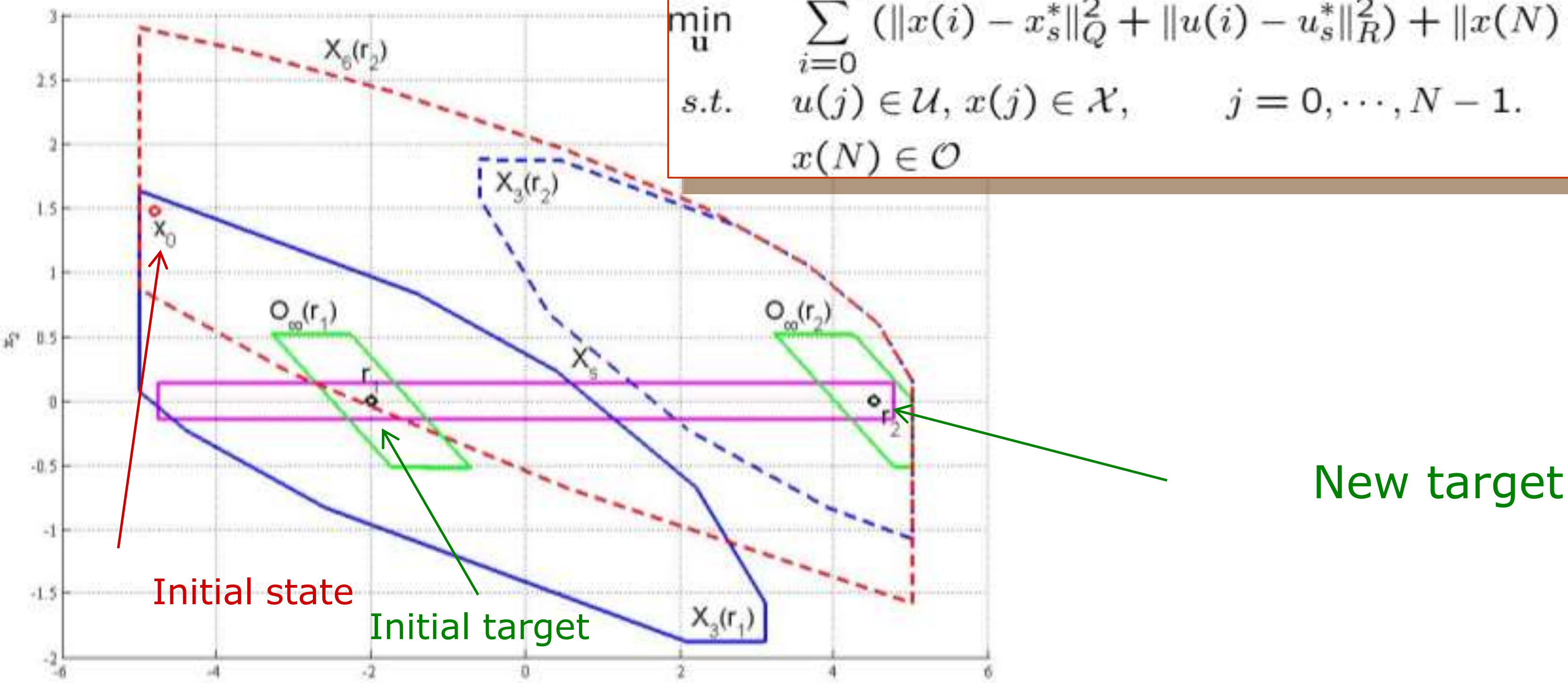
- *Market-driven production*
  - Production goals
  - Cost functions
  - Operation constraints
- *Real Time Optimization*
  - Disturbances
  - Estimation errors
  - Model parameters

Frequent target changes

Stability issues

- Stability loss:
  - Redesign of the terminal conditions
- Feasibility loss:

$$
\begin{aligned}
\min_{\mathbf{u}} \quad & \sum_{i=0}^{N-1} (\|x(i)-x_s^*\|_Q^2 + \|u(i)-u_s^*\|_R^2) + \|x(N)-x_s^*\|_P^2 \\
s.t. \quad & u(j) \in \mathcal{U},\ x(j) \in \mathcal{X}, \qquad j = 0, \cdots, N-1. \\
& x(N) \in \mathcal{O}
\end{aligned}
$$



MPC for tracking

$$
\begin{aligned}
\min_{\mathbf{u}} \quad & \sum_{i=0}^{N-1} \ell(x(i)-x_s^*, u(i)-u_s^*) + V_f(x(N)-x_s^*) \\
s.t. \quad & x(0) = x \\
& x(j+1) = f(x(j), u(j)) \\
& u(j) \in \mathcal{U},\ x(j) \in \mathcal{X}, \qquad j = 0, \cdots, N-1. \\
& x(N) - x_s^* \in X_f
\end{aligned}
$$

MPC for regulation

$$
\begin{aligned}
\min_{\mathbf{u}, y_s} \quad & \sum_{i=0}^{N-1} \ell(x(i)-x_s, u(i)-u_s) + V_f(x(N)-x_s) + V_O(y_s - y_t) \\
s.t. \quad & x(0) = x \\
& x(j+1) = f(x(j), u(j)) \\
& x_s = f(x_s, u_s),\ y_s = h(x_s, u_s) \\
& u(j) \in \mathcal{U},\ x(j) \in \mathcal{X}, \quad j = 0, \cdots, N-1. \\
& (x(N), y_s) \in \Gamma.
\end{aligned}
$$

MPC for tracking

(Limon et al 2008)

- Artificial set-point (x_s, u_s) as decision variables
- Offset cost function $V_O(y_s - y_t)$
- Extended terminal constraint

MPC for tracking

Example:

Consider the discrete time LTI system:

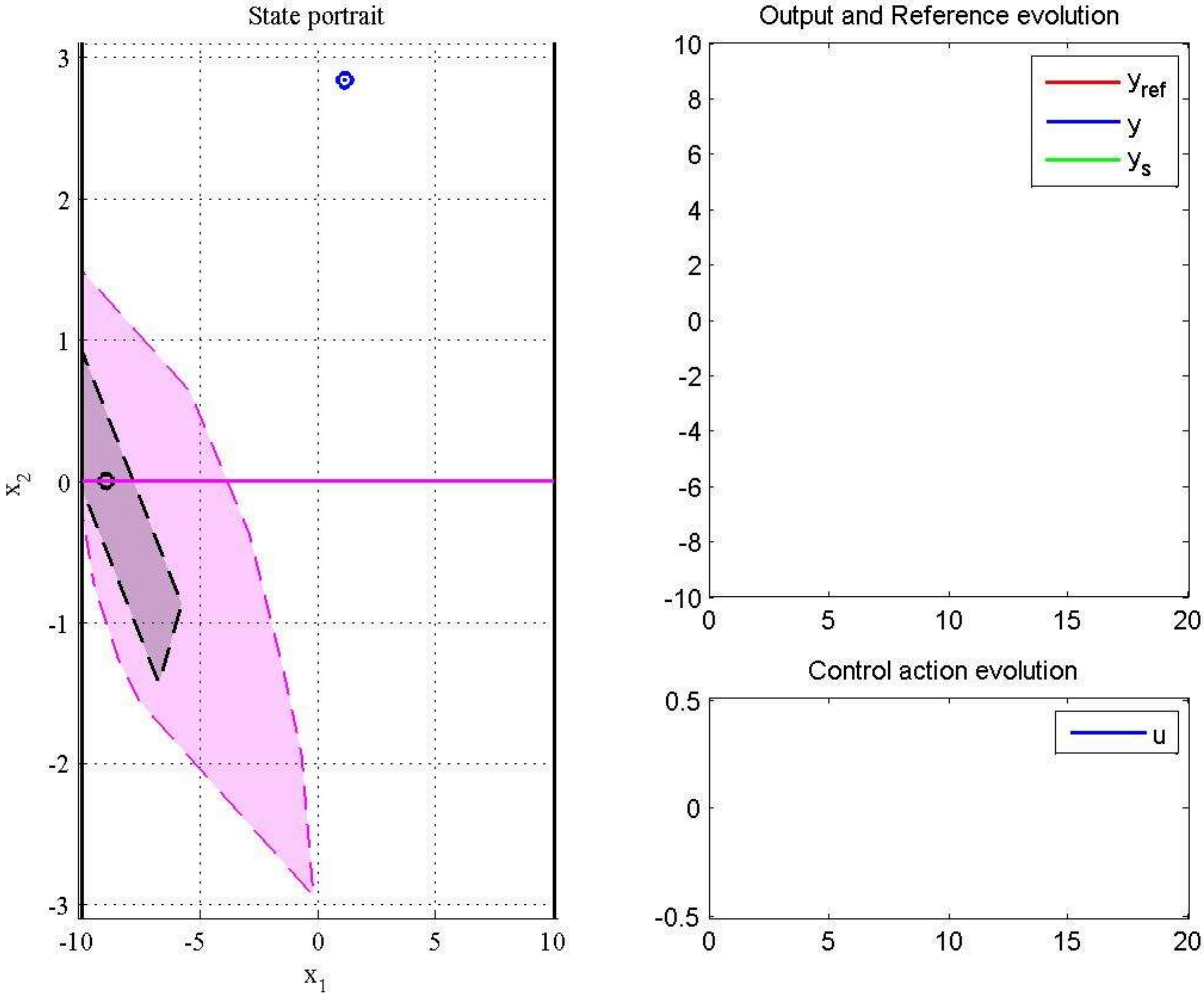
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Subject to the following hard constraints:

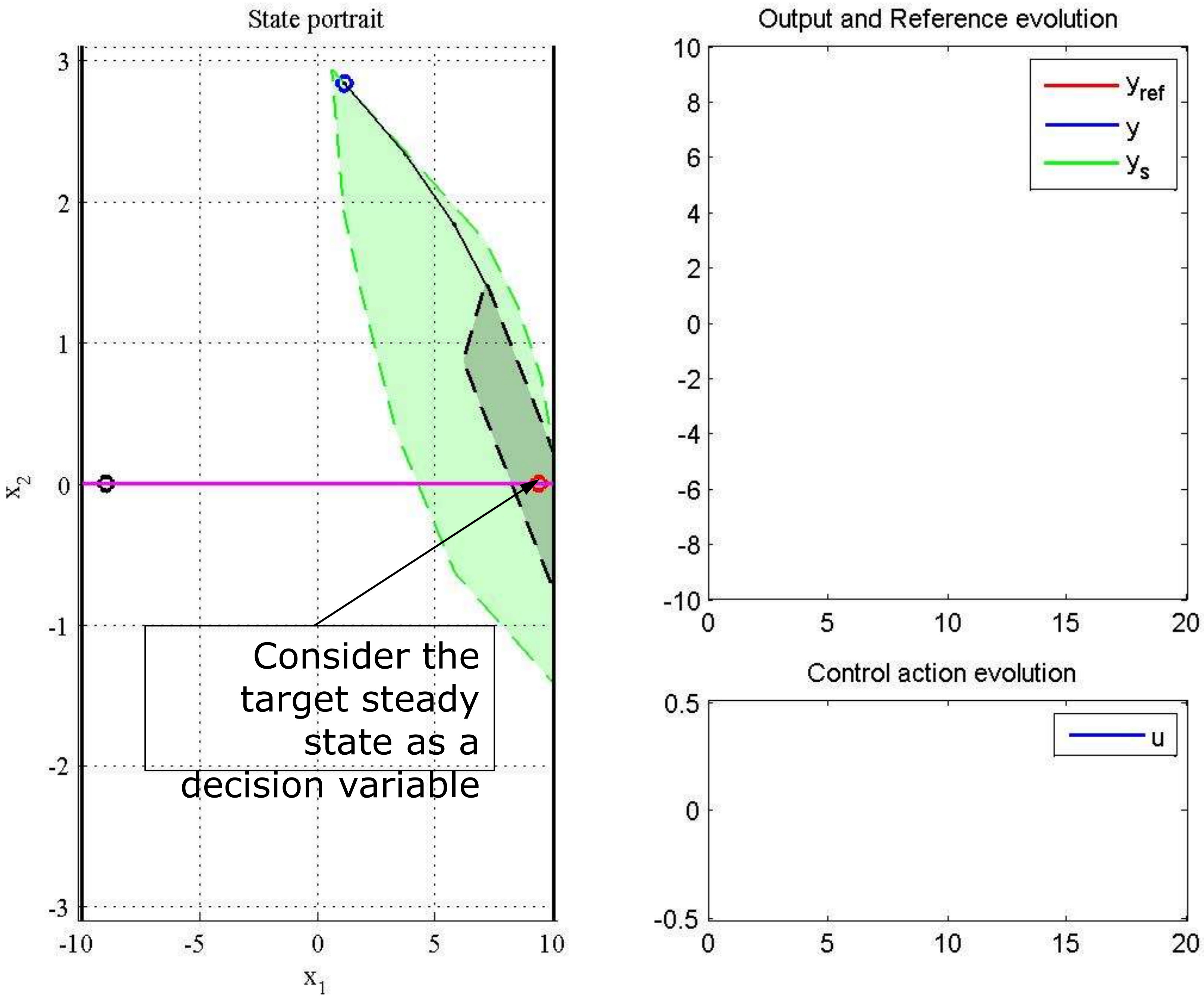
$$\begin{aligned} \mathcal{U} &= \{u \in \mathbb{R} : |u| \leq 0.5\} \\ \mathcal{X} &= \{x \in \mathbb{R}^2 : |x_1| \leq 10,\ |x_2| \leq 4\} \end{aligned}$$

Controller parameters:

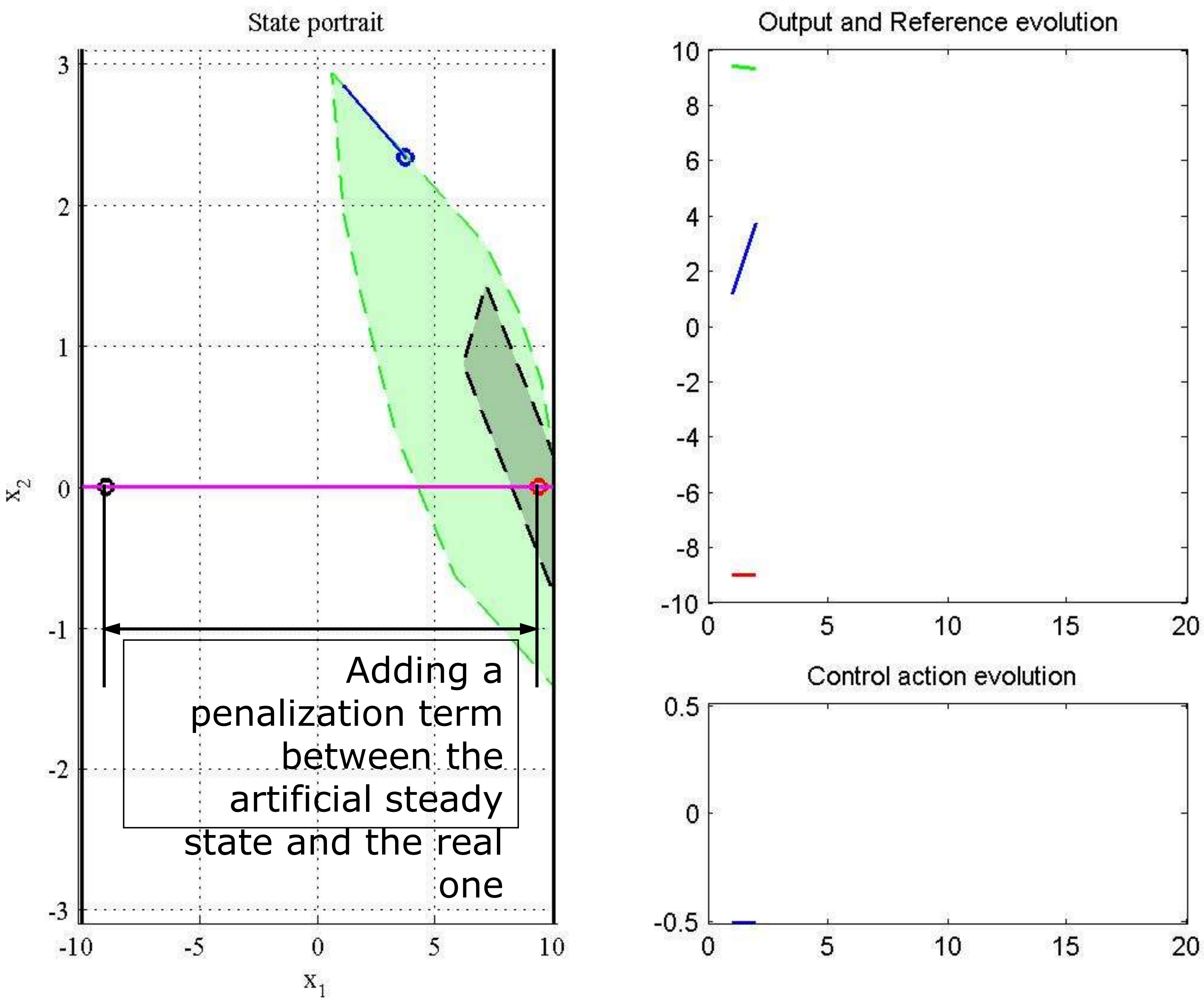
$$\begin{aligned} &N = 3 \\ &\ell(x,u) = \|x\|^2 + \|u\|^2 \\ &V_f(x) = \|x\|_P^2 \\ &V_O(y) = \|y\|_T^2 \end{aligned}$$

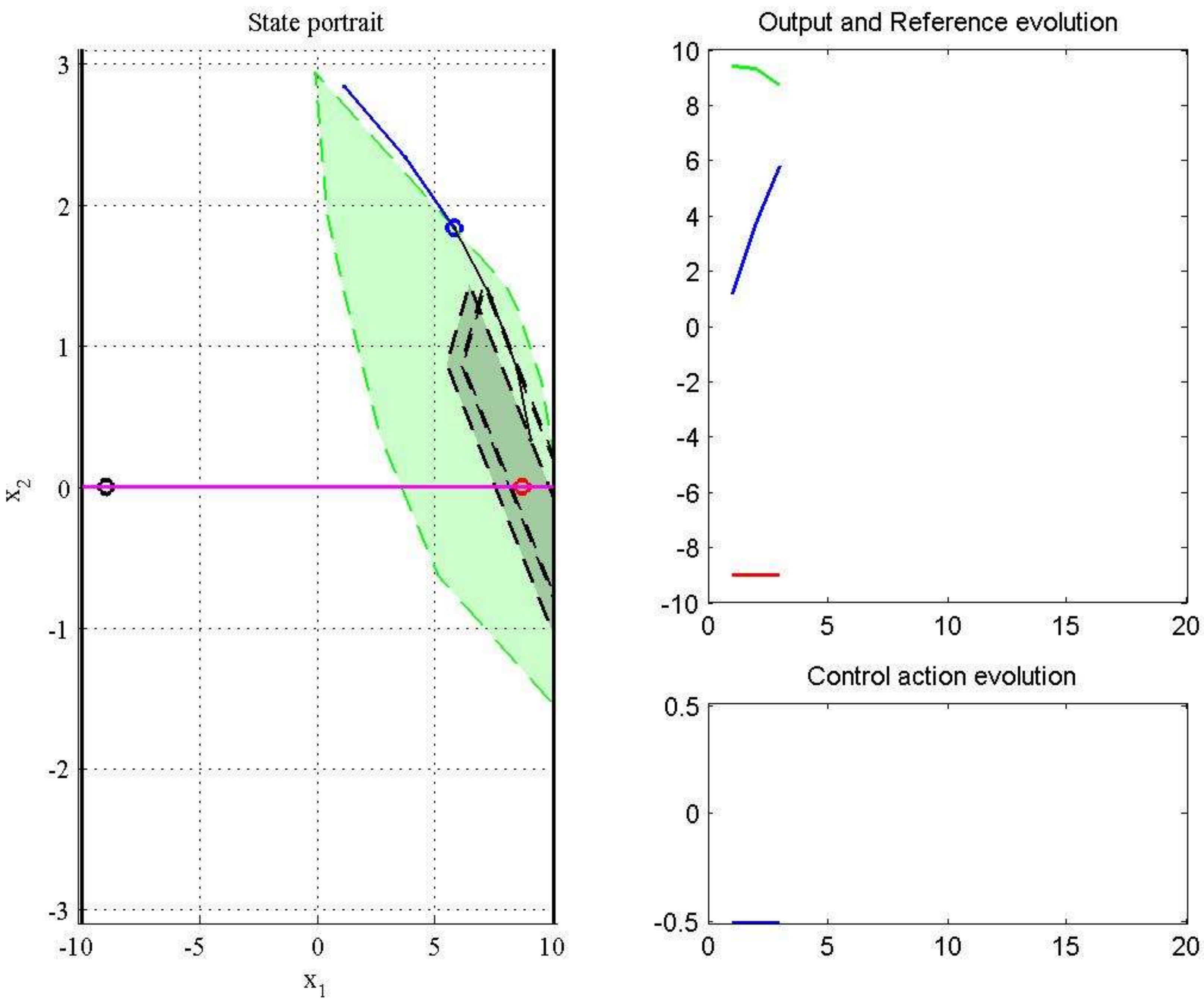


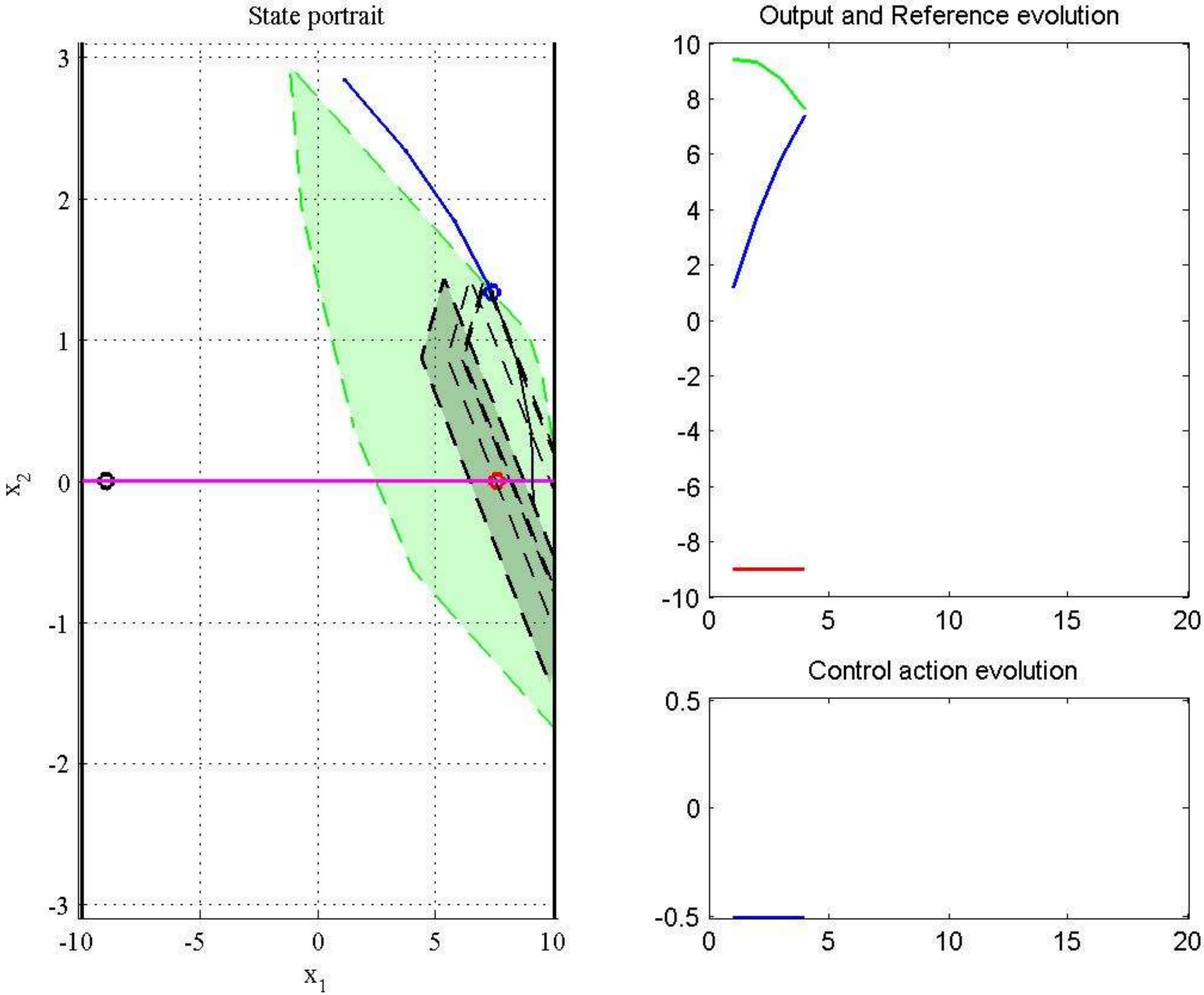
State portrait
x_2
x_1
Output and Reference evolution
y_{ref}
y
y_s
Control action evolution
u



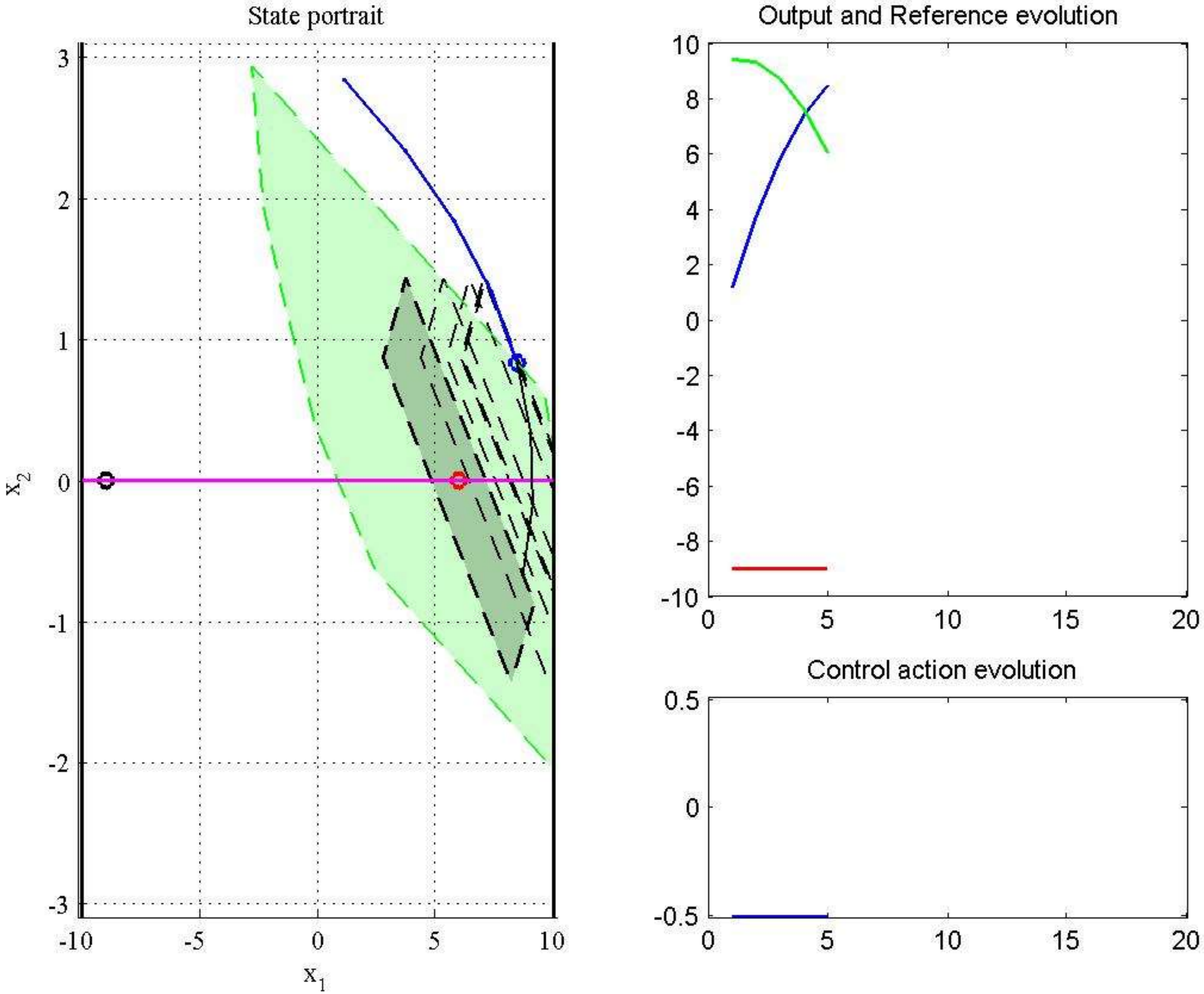
State portrait
Output and Reference evolution
Control action evolution
Consider the target steady state as a decision variable
x_1
x_2
y_{ref}
y
y_s
u

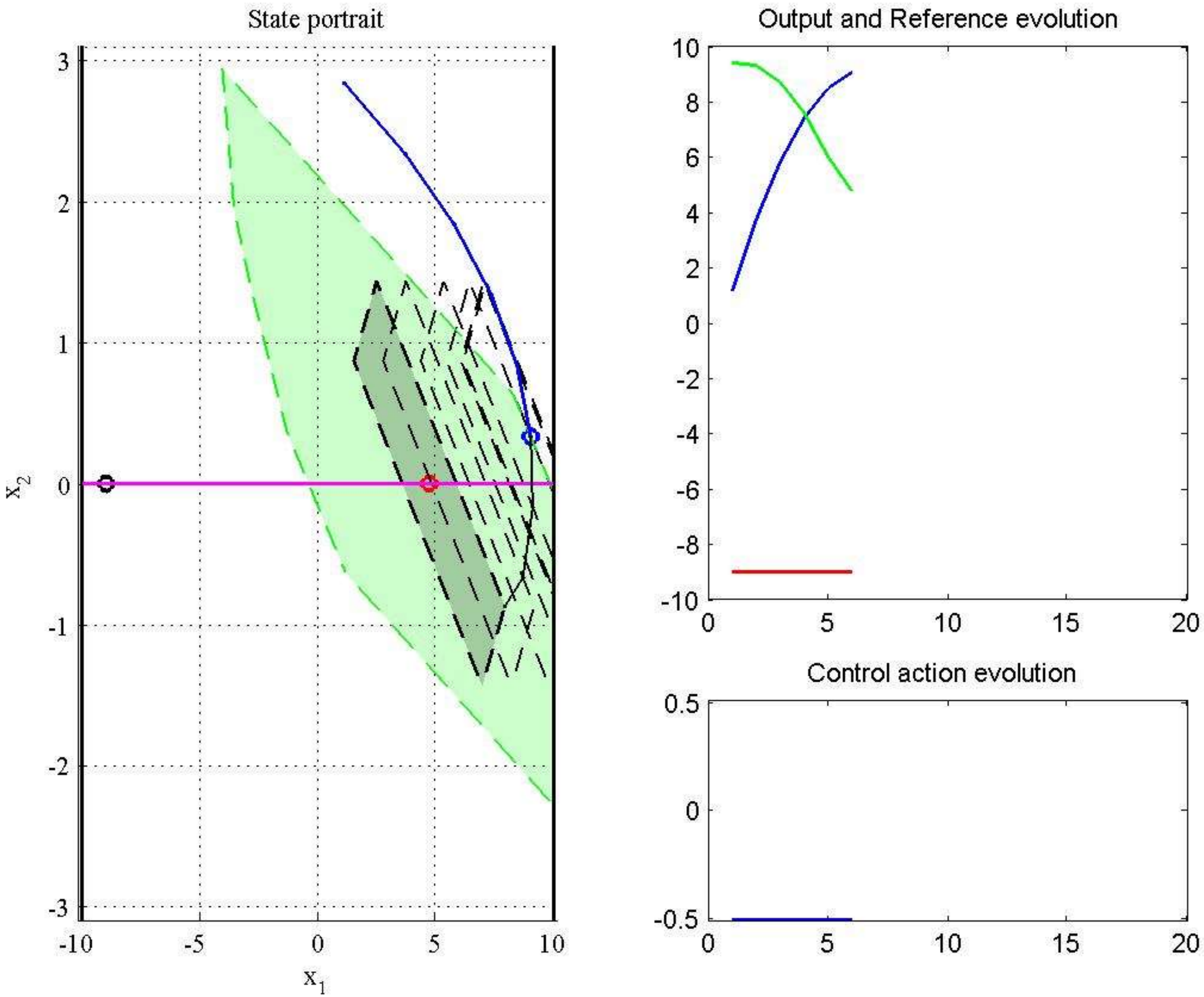

State portrait
x_2
x_1
Adding a penalization term between the artificial steady state and the real one
Output and Reference evolution
Control action evolution


State portrait
Output and Reference evolution
Control action evolution
x_1
x_2

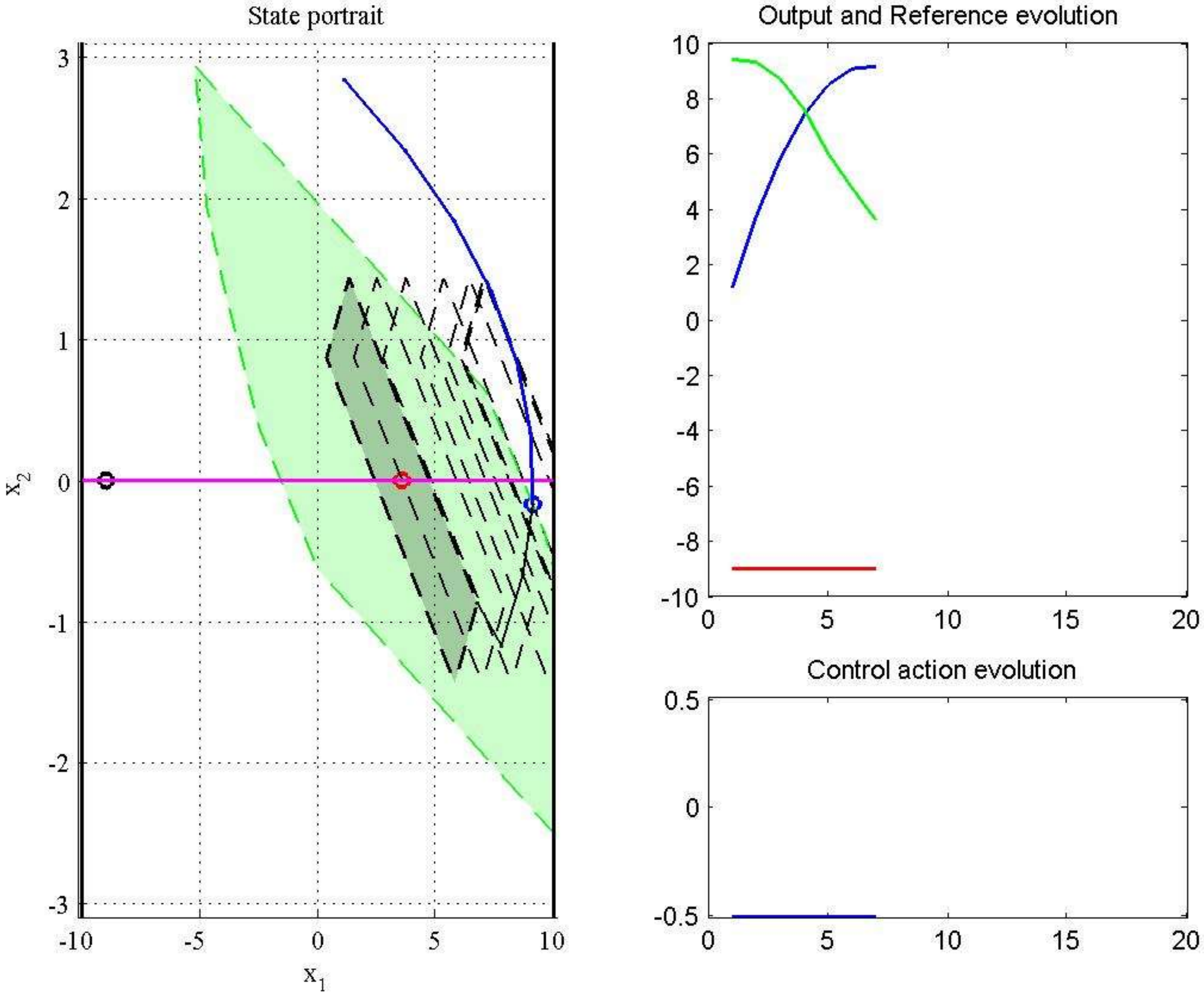


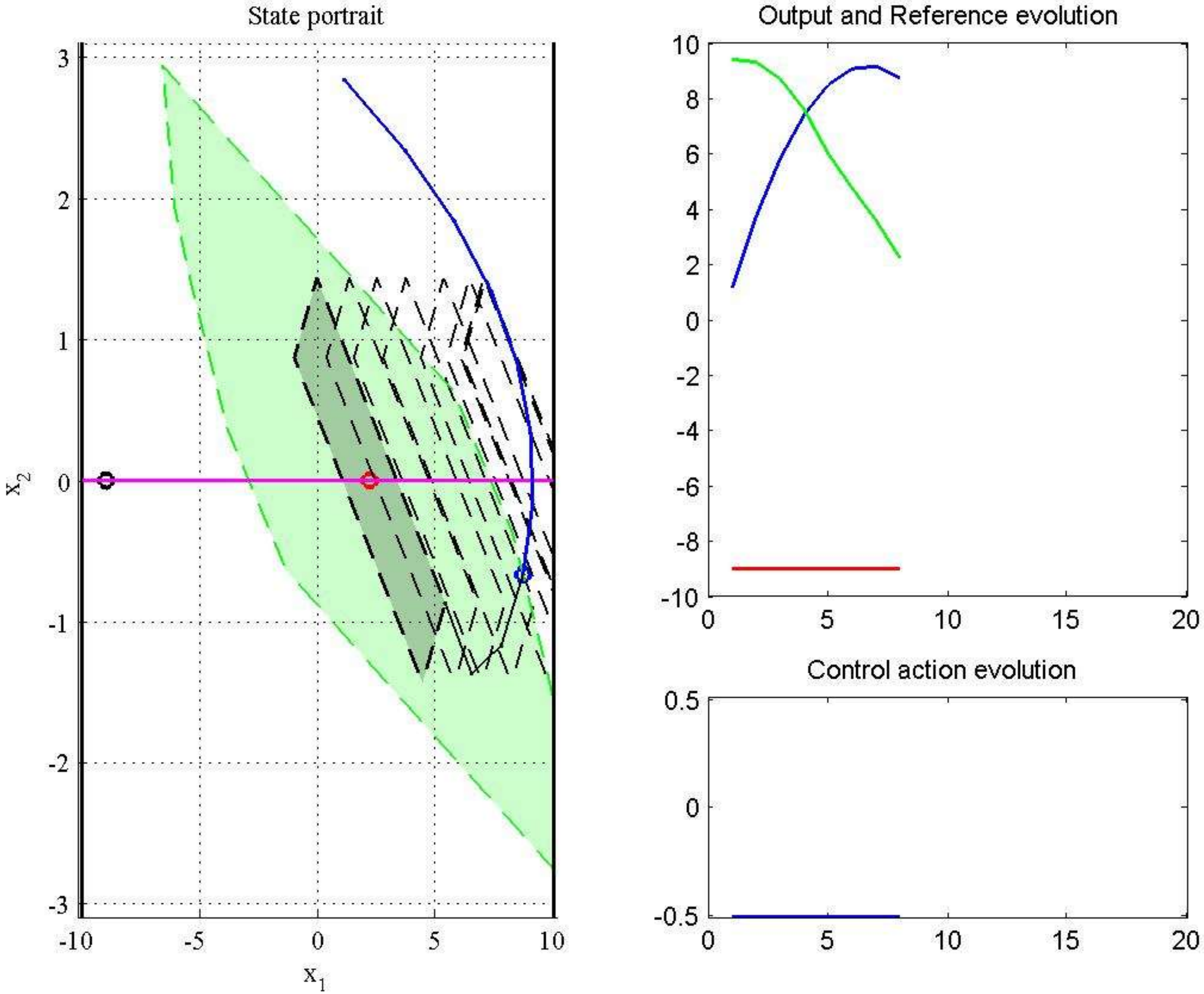
State portrait
x2
x1
Output and Reference evolution
Control action evolution


State portrait
x2
x1
Output and Reference evolution
Control action evolution

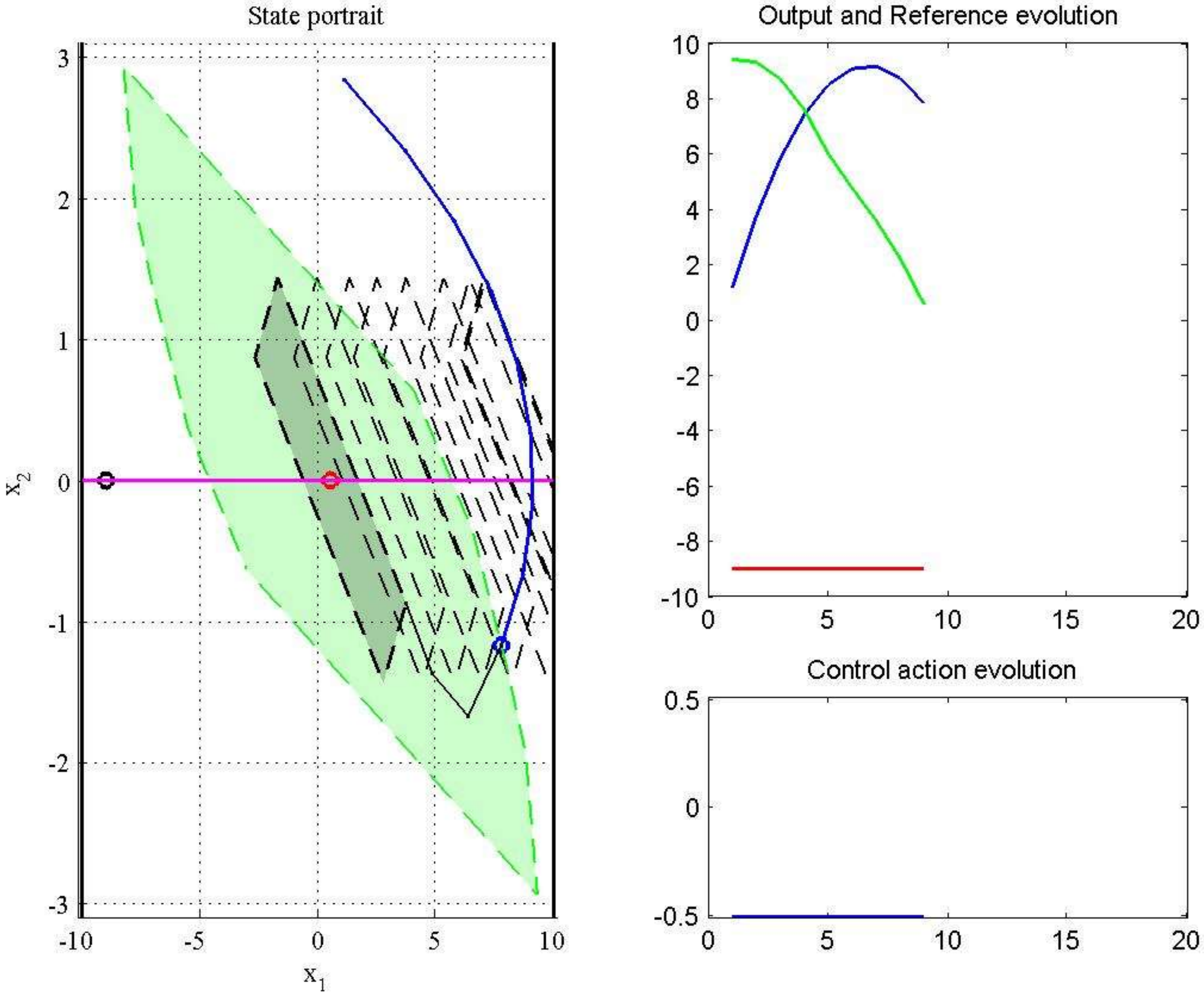


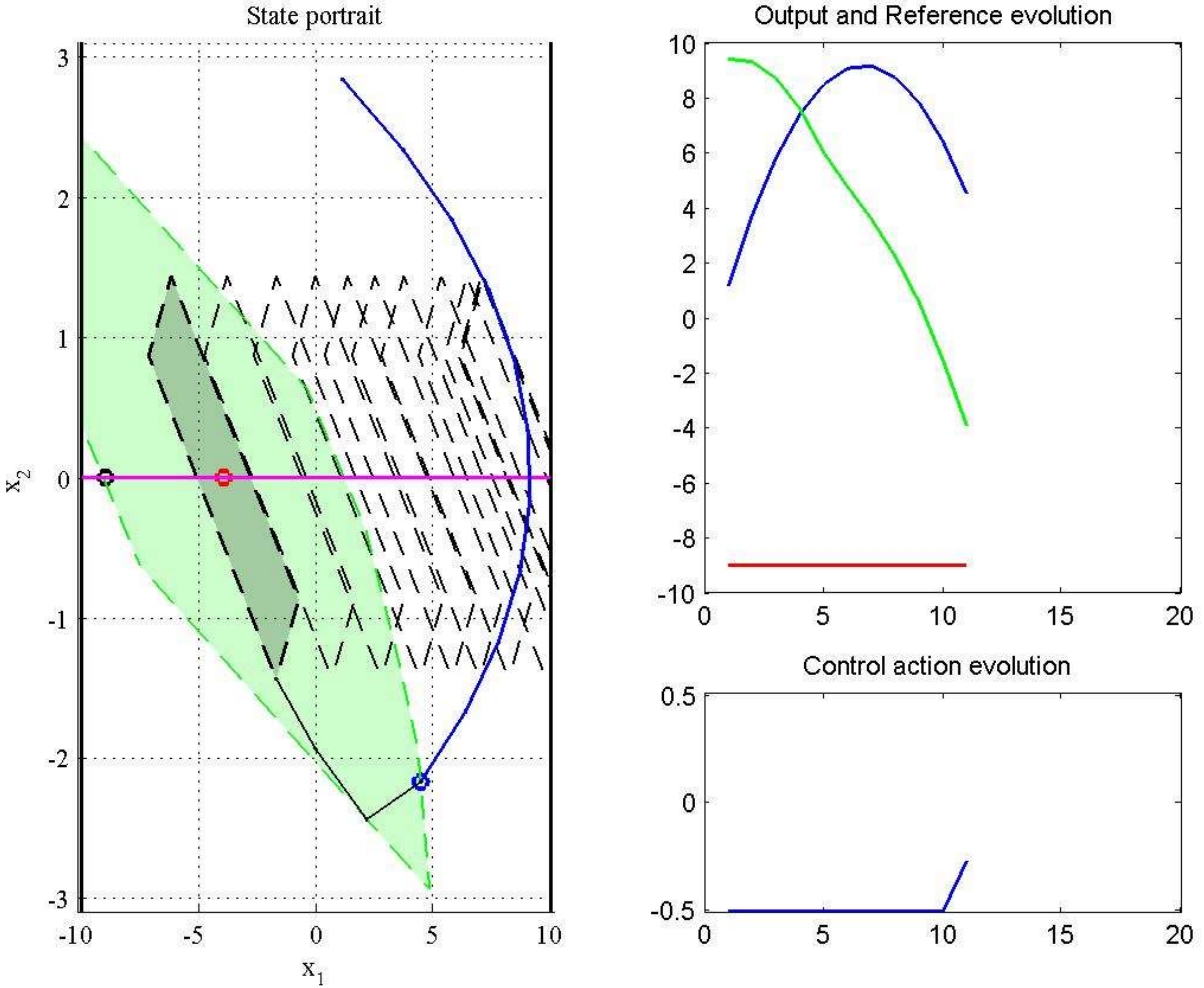
State portrait
Output and Reference evolution
Control action evolution
x1
x2

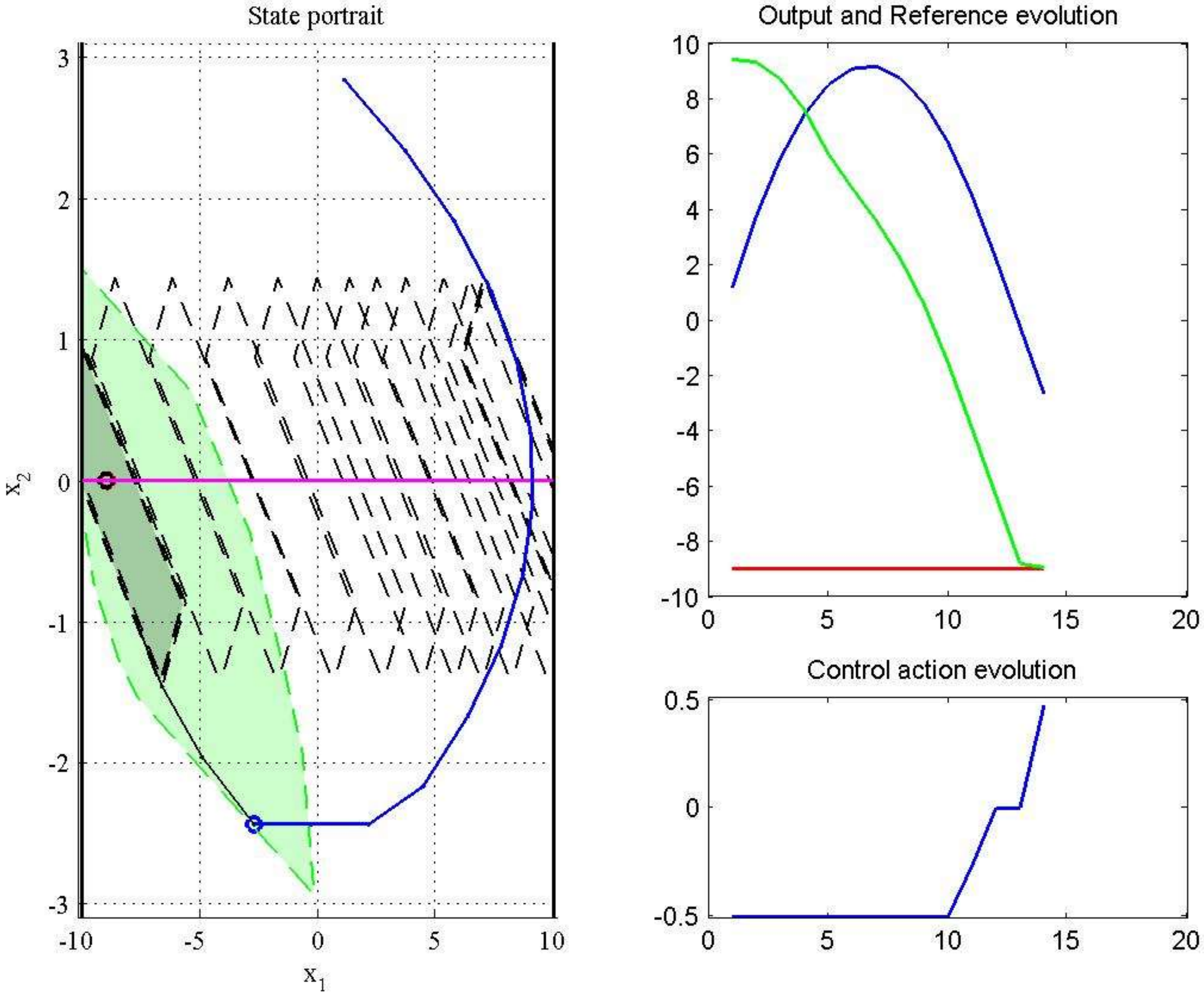

State portrait
Output and Reference evolution
Control action evolution
x1
x2

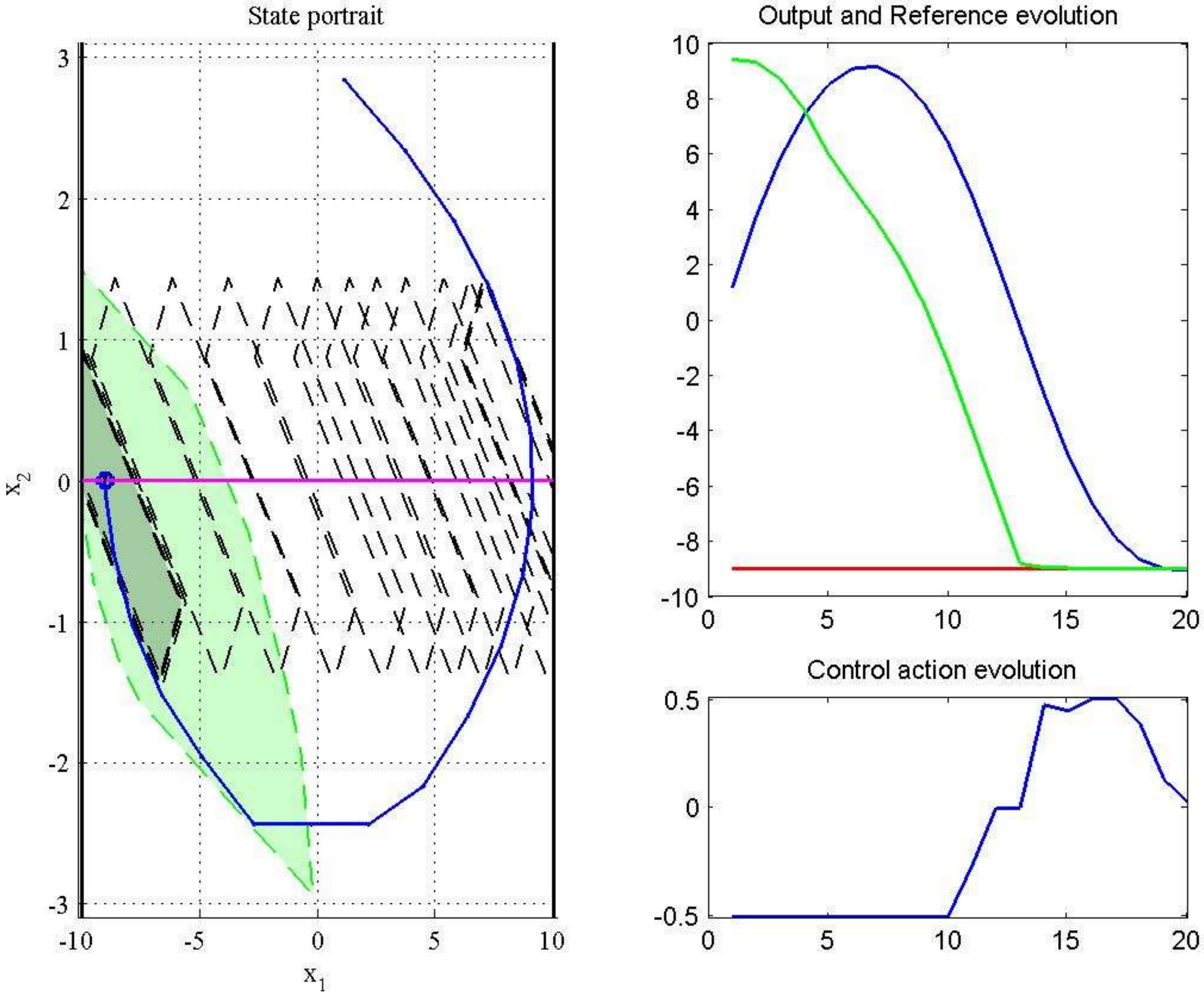


State portrait
x_2
x_1
Output and Reference evolution
Control action evolution

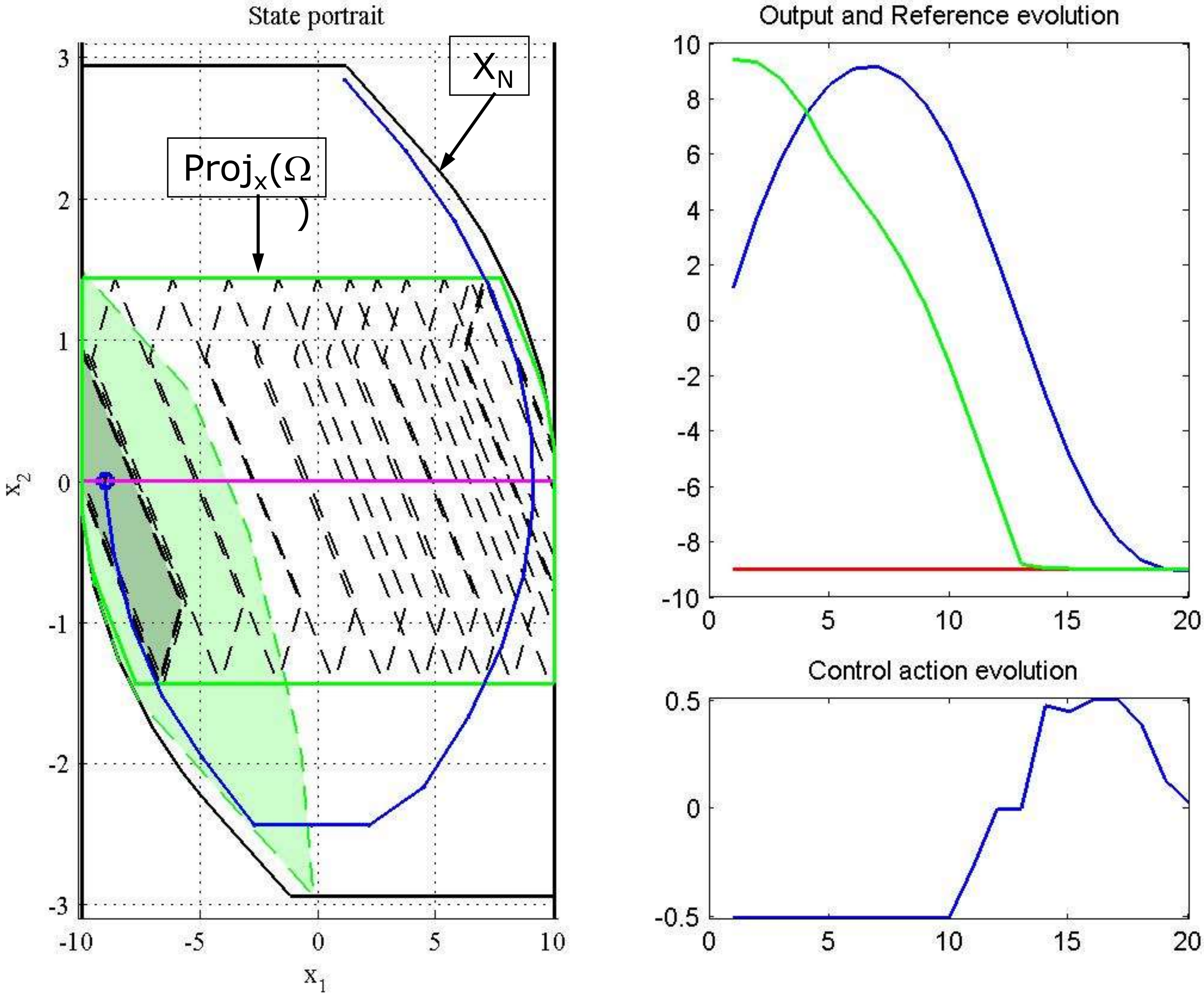

State portrait
x_2
x_1
Output and Reference evolution
Control action evolution


State portrait
Output and Reference evolution
Control action evolution
x2
x1
-10
-5
0
5
10
3
2
1
0
-1
-2
-3
10
8
6
4
2
0
-2
-4
-6
-8
-10
0
5
10
15
20
0.5
0
-0.5
0
5
10
15
20


State portrait
x1
x2
Output and Reference evolution
Control action evolution



State portrait
Output and Reference evolution
Control action evolution
x_2
x_1


State portrait
X_N
Proj_x(Ω)
x_2
x_1
Output and Reference evolution
Control action evolution

MPC for tracking

- The steady output y_s univocally characterizes an equilibrium point

$$\begin{aligned}
\min_{\mathbf{u},y_s} \quad & \sum_{i=0}^{N-1} \ell(x(i)-x_s, u(i)-u_s) + \boxed{V_f(x(N)-x_s)} + \boxed{V_O(y_s-y_t)} \\
s.t. \quad & x(0) = x \\
& x(j+1) = f(x(j), u(j)) \\
& (x(j), u(j)) \in Z, \quad j = 0, \cdots, N-1. \\
\\
& x_s = f(x_s, u_s), y_s = h(x_s, u_s) \\
& (x_s), u_s) \in Z \\
\\
& \boxed{(x(N), y_s) \in \Gamma.}
\end{aligned}$$

Offset cost function

Terminal cost function

Extended terminal constraint

Stabilizing design

- The steady output y_s univocally characterizes an equilibrium point
- There exists a control law $u = \kappa(x, y_s)$ such that$\forall (x, y_s) \in \Gamma$,
  - $(x, \kappa(x, y_s)) \in \mathcal{Z}$, $y_s \in \mathcal{Y}_s$, and $(x^+, y_s) \in \Gamma$
  - $V_f(x, y_s)$ is a Lyapunov function such that

$$V_f(x^+ - x_s, y_s) - V_f(x - x_s, y_s) \leq -\ell(x - x_s, \kappa(x, y_s) - u_s)$$

  where $x^+ = f(x, \kappa(x, y_s))$

- A sensible design: equality constraint
  - Terminal cost function: $V_f(x) = 0$
  - Terminal constraint: $x(N) = x_s$

Stability theorem

Theorem

Let the offset cost function $V_O(y)$ be a positive definite convex function and let the set $\mathcal{Y}_s$ be a convex set.
Let $\mathcal{X}_N$ be the feasibility region of $P_N(x, y_t)$ and let $N \geq n$.
Then $\forall x_0 \in \mathcal{X}_N$ and $\forall\, y_t$,

- the controlled system is asymptotically stable at an equilibrium point,
- fulfils the constraints throughout its evolution and
- converges to an equilibrium point such that

$$y_s^* = \arg \min_{y_s \in \mathcal{Y}_s} V_O(y_s - y_t)$$

Properties of the controller

- Stability for any change of the target

- Continuous Stirred Tank Reactor (CSTR)
  - Plant model *[Magni'01]*:

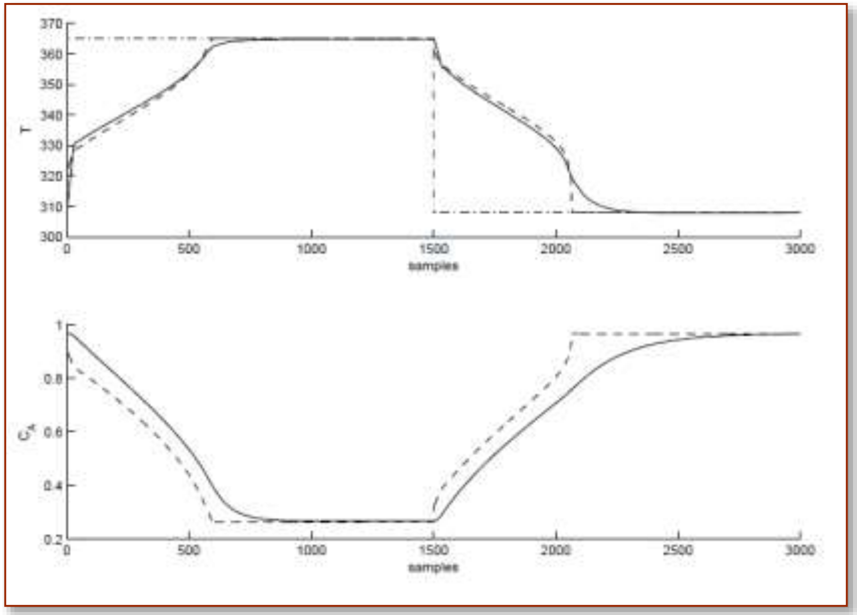
$$\dot{C}_A = \frac{q}{V}(C_{Af} - C_A) - k_o e^{(\frac{-E}{RT})} C_A$$
$$\dot{T} = \frac{q}{V}(T_f - T) - \frac{\Delta H}{\rho C_p} k_o e^{(\frac{-E}{RT})} C_A + \frac{UA}{V\rho C_p}(T_c - T)$$

Controlled output T

  - Constraints:

$$\begin{array}{ccccc} 0 & \leq & C_A[mol/l] & \leq & 1 \\ 280 & \leq & T[K] & \leq & 370 \\ 280 & \leq & T_c[K] & \leq & 370 \end{array}$$

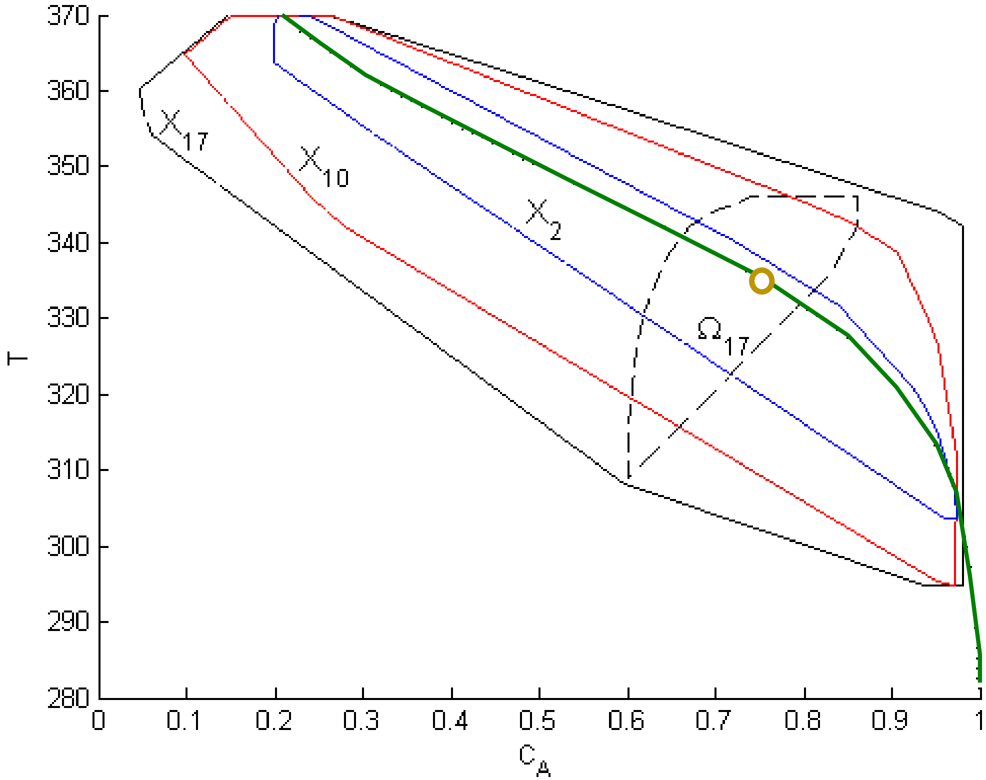
  - $\mathcal{Y}_s = [304.17, 370]$
  - sampling time=0.03 min.
  - $\mathrm{N} = 2$



Properties of the controller

- Larger domain of attraction

CSTR



Equilibrium points

DoA of MPC for regulation (N=17)

DoA of MPC for tracking (N=2)

DoA of MPC for tracking (N=10)

DoA of MPC for tracking (N=17)

$$\mathcal{X}_s \subseteq X_N$$

Controller optimality

- The solutions of MPC and MPC for tracking may differ (when both are feasible)
  - Reason: the artificial reference

- Then the local optimality property of MPC may be lost

- This can be solved by taking

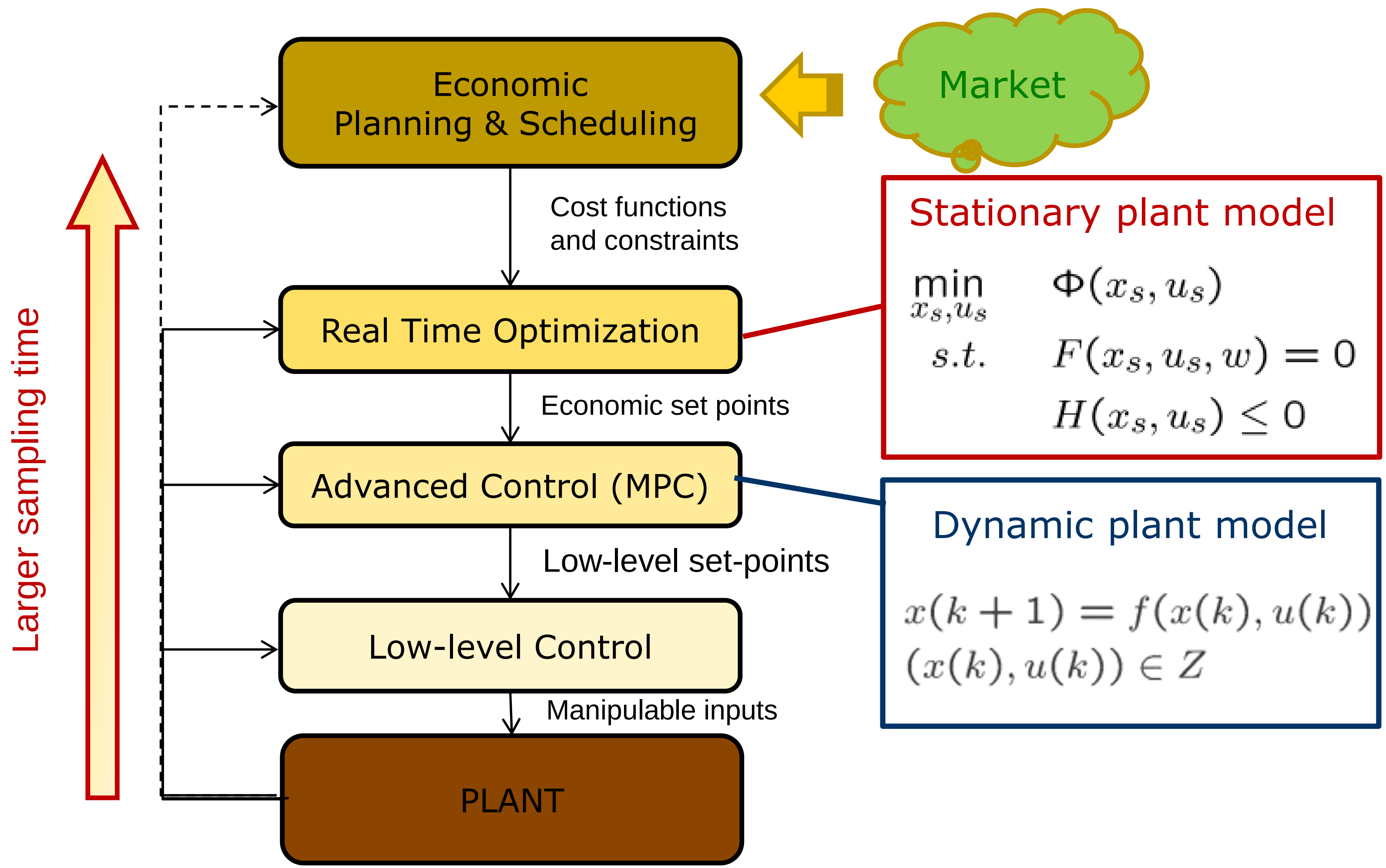
$$V_O(y - yt) \geq c\|y - y_t\|$$

For all $c \geq c^*$

Outline

- Stabilizing design of predictive controllers
- Robustness and robust design
- Set-point tracking predictive control
- **Economic predictive control**
- Conclusions

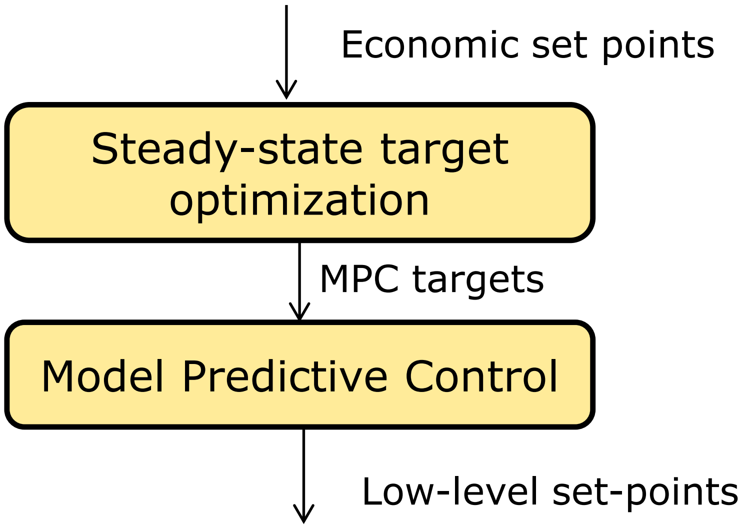
Optimal operation of a plant



Model Predictive Control

- RTO disadvantages *[Engell, 2007]*
  - Slow reaction to process variations
  - Mismatches between the models of RTO and MPC

The RTO may provide inconsistent set points to the MPC



Two layer MPC

Steady-state Target Optimization

- A local approximation of the profit function Φ must be provided by the RTO

$$\ell_{eco}(y_s - y_t; p)$$

- The *optimal* steady state consistent with the prediction model of the MPC is computed

$$\begin{aligned} (x_s^*, u_s^*) = \arg\min_{x_s, u_s} \quad & \ell_{eco}(y_s - y_t; p) \\ s.t. \quad & x_s = f(x_s, u_s) \\ & (x_s, u_s) \in Z \end{aligned}$$

- The optimal target for the MPC (x_s^*, u_s^*) is provided.

Model Predictive Control

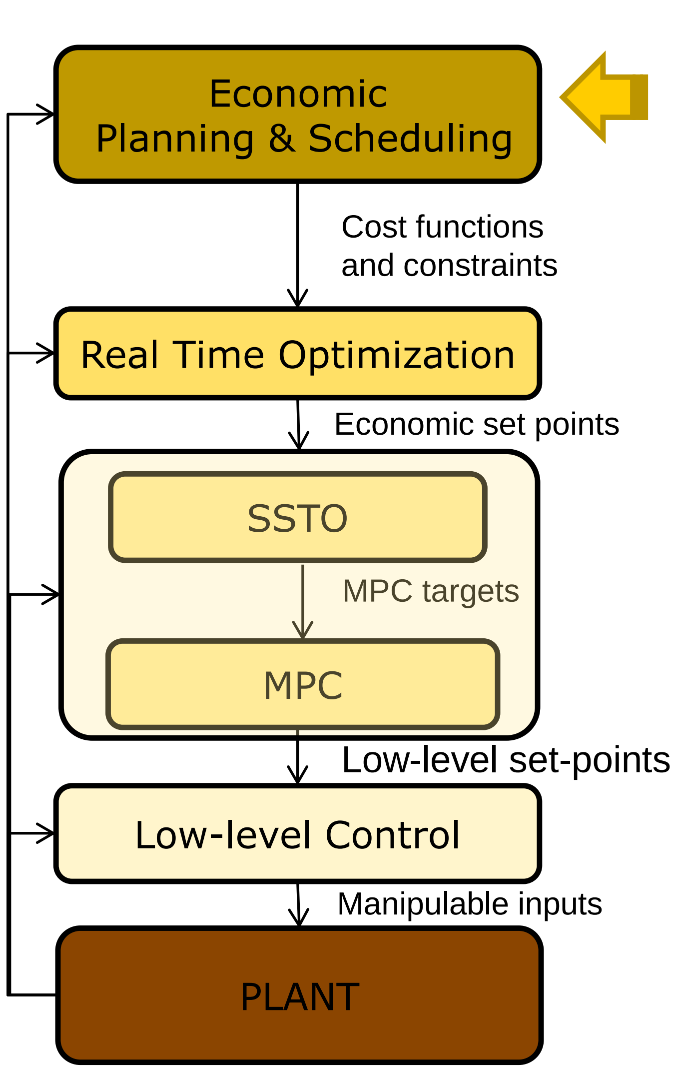
- Objective: regulate the system to the MPC target

$$\begin{aligned} \min_{\mathbf{u}} \quad & V_N(x,\mathbf{u}) = \sum_{i=0}^{N-1} \ell(x(i)-x_s^*, u(i)-u_s^*) + V_f(x(N)-x_s^*) \\ s.t. \quad & x(i+1) = f(x(i),u(i)) \\ & x(0) = x \\ & u(j) \in \mathcal{U},\ x(j) \in \mathcal{X}, \qquad j = 0,\cdots,N-1. \\ & x(N) - x_s^* \in \Omega \end{aligned}$$

  - $\ell(x-x_s^*, u-u_s^*)$ measures the traking error
- The optimal predicted sequence $\mathbf{u}^*(x)$ is computed
- Receding horizon

$$\kappa_N(x) = \mathbf{u}^*(0;x)$$

Economic optimization

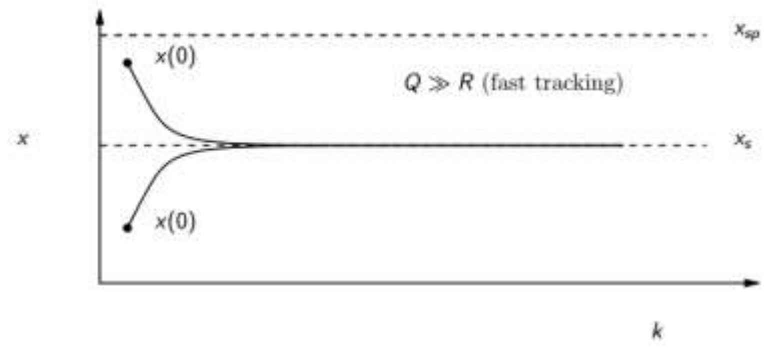


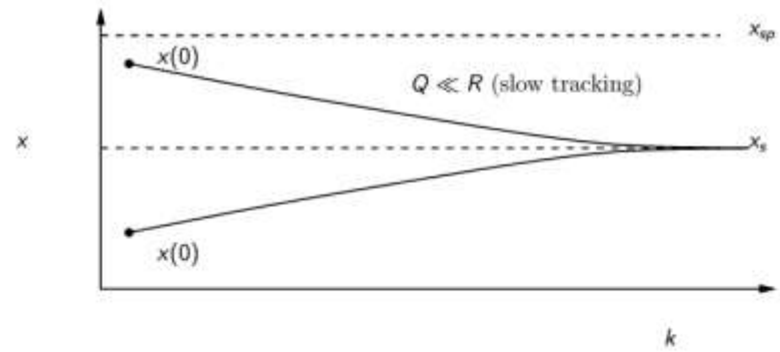
- Economic optimization of the steady operation (Economic set point)
- The transient control problem is posed as a tracking control problem to the MPC target

Is this the economically optimal operation of the plant?

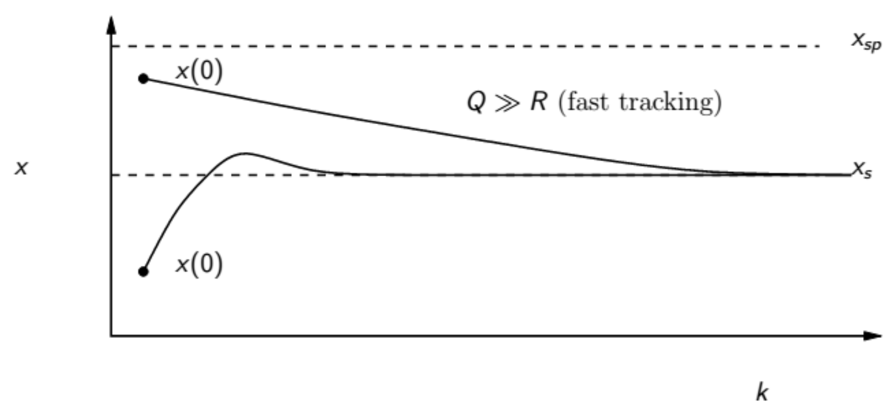
Economic optimization

- Tracking control to the MPC target





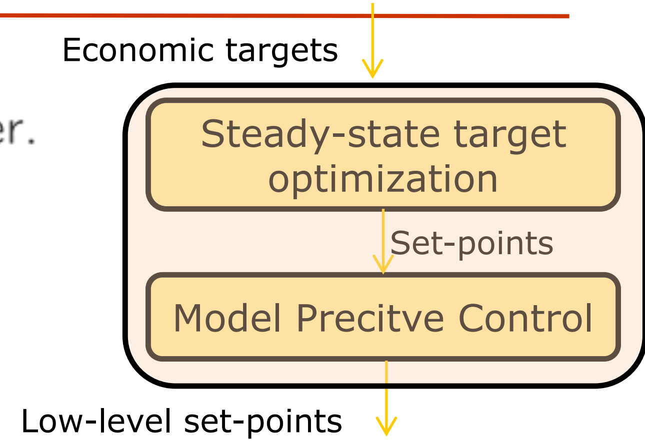
- Economic operation of the plant



The economic cost function should be used to measure the performance of the transient

Integration of SSTO in the MPC

- The two-layer MPC is integrated in a single layer.
- This generalizes the one-layer MPC *[Zanin'02]*.
- Better closed-loop performance



Idea: use $\ell_{eco}(\cdot)$ as offset cost function in the MPC for tracking

$$\begin{aligned}
\min_{\mathbf{u}, y_s} \quad & \sum_{i=0}^{N-1} \ell(x(i) - x_s, u(i) - u_s) + V_f(x(N) - x_s) + \ell_{eco}(y_s - y_t) \\
s.t. \quad & x(0) = x \\
& x(j+1) = f(x(j), u(j)) \\
& u(j) \in \mathcal{U}, \; x(j) \in \mathcal{X}, \quad j = 0, \cdots, N-1. \\
& x_s = f(x_s, u_s), \; y_s = h(x_s, u_s) \\
& u_s \in \mathcal{U}, \; x_s \in \mathcal{X}, \\
& x(N) = x_s
\end{aligned}$$

Economic MPC

- MPC for regulation to the setpoint

$$
\begin{aligned}
\min_{\mathbf{u}} \quad & V_N(x,\mathbf{u}) = \sum_{i=0}^{N-1} \ell(x(i)-x_s^*, u(i)-u_s^*) \\
s.t. \quad & u(j) \in \mathcal{U},\ x(j) \in \mathcal{X}, \qquad j = 0,\cdots,N-1. \\
& x(N) = x_s^*
\end{aligned}
$$

- Economic MPC

$$
\begin{aligned}
\min_{\mathbf{u}} \quad & V_N(x,\mathbf{u}) = \sum_{i=0}^{N-1} \ell_{eco}(y(i)-y_t; p) \\
s.t. \quad & u(j) \in \mathcal{U},\ x(j) \in \mathcal{X}, \qquad j = 0,\cdots,N-1. \\
& x(N) = x_s^*
\end{aligned}
$$

  - The economic stage cost function is not positive definite
  - Existing Lyapunov stability results can not be used
  - **Stability issues**

Economic MPC

- Dissipativity condition : there exists a storage function $\lambda(\cdot)$

$$\lambda(f(x,u)) - \lambda(x) \leq \left\{\ell_{eco}(y - y_t) - \ell_{eco}(y_s^* - y_t)\right\} - \rho(\|x - x_s^*\|)$$

  - For linear systems and convex economic cost function, $\lambda(\cdot)$ exists.

- Idea of the stability proof *(Angeli'12)*:
  - Consider the rotated cost:

$$L(x,u) = \ell_{eco}(y - y_t) - \ell_{eco}(y_s^* - y_t) + \lambda(x) - \lambda(f(x,u))$$

  - $L(x,u) \geq \rho(\|x - x_s^*\|)$
  - The MPC control law with the rotated cost $L(x,u)$ is equal to the economic MPC control law

Economic MPC for changing targets

- If the economic set point changes, feasibility may be lost

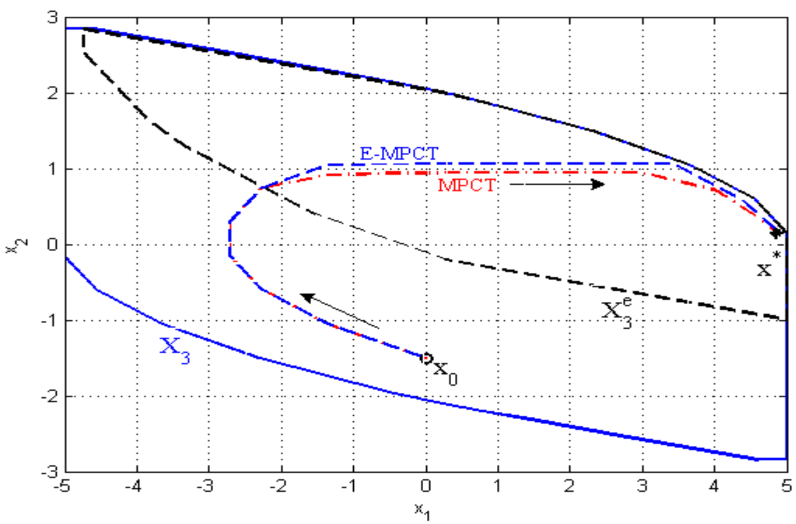
- **Motivation:** guaranteed feasibility (MPC for tracking) + economic optimality (Economic MPC)

- **MPC problem**:

$$
\begin{aligned}
\min_{\mathbf{u},y_s} \quad & \sum_{j=0}^{N-1} \ell_{eco}(y(j) - y_t + y_s - y_s^*; p) + c\|y_s - y_s^*\| \\
s.t. \quad & x(0) = x, \\
& x(j+1) = f(x(j), u(j)), \quad j=0,\cdots,N-1 \\
& (x(j), u(j)) \in \mathcal{Z}, \quad j=0,\cdots,N-1 \\
& x_s = f(x_s, u_s), y_s = h(x_s, u_s) \\
& (x_s, u_s) \in \mathcal{Z} \\
& x(N) = x_s
\end{aligned}
$$

Properties

- Feasibility for any changing economic criterion.
- Domain of attraction larger than the economic MPC.
- Economic Optimality: there exists a c^* such that for $c \geq c^*$, the proposed controller provides the solution of the economic MPC.



$Q = I_2 \quad R = I_2$

$x_t = (6, 3) \quad x^* = (5; 0.15)$

$$\Phi = \sum_{k=0}^{T} |x(k) - x_t|_Q^2 + |u(k) - u_t|_R^2 - (|x^* - x_t|_Q^2 + |u^* - u_t|_R^2)$$

Measure	E-MPC	MPCT	E-MPCT
Φ	226.7878	304.3342	226.7878

- Economic MPC: $N = 10$
- Economic MPC for tracking: $N = 3$

Example: the double integrator

Double integrator

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0.5 \\ 1 & 0.5 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Constraints:

$$\mathcal{U} = \{u \in \mathbb{R}^2 : |u| \leq 0.3\}$$
$$\mathcal{X} = \{x \in \mathbb{R}^2 : |x| \leq 5\}$$

Cost functions:

Economic MPC (E-MPC):

$$V_N^e(x;\mathbf{u}) = \sum_{i=0}^{N-1} |x(i) - x_t|_Q^2 + |u(i) - u_t|_R^2$$

MPC for tracking (MPCT):

$$V_N^t(x;\mathbf{u}) = \sum_{i=0}^{N-1} |x(i) - x_s|_Q^2 + |u(i) - u_s|_R^2 + V_O(x_s - x_t)$$

Economic MPC for changing targets (E-MPCT):

$$V_N(x;\mathbf{u}) = \sum_{i=0}^{N-1} |x(i) - x_t + x_s - x_s^*|_Q^2 + |u(i) - u_t + u_s - u_s^*|_R^2 + V_O(x_s - x_s^*)$$

Controller parameters:

$$Q = I_2 \quad R = I_2$$
$$x_t = (6,3) \quad x^* = (5;0.15)$$

Example: the double integrator

- **Economic optimality:**
  - N=10

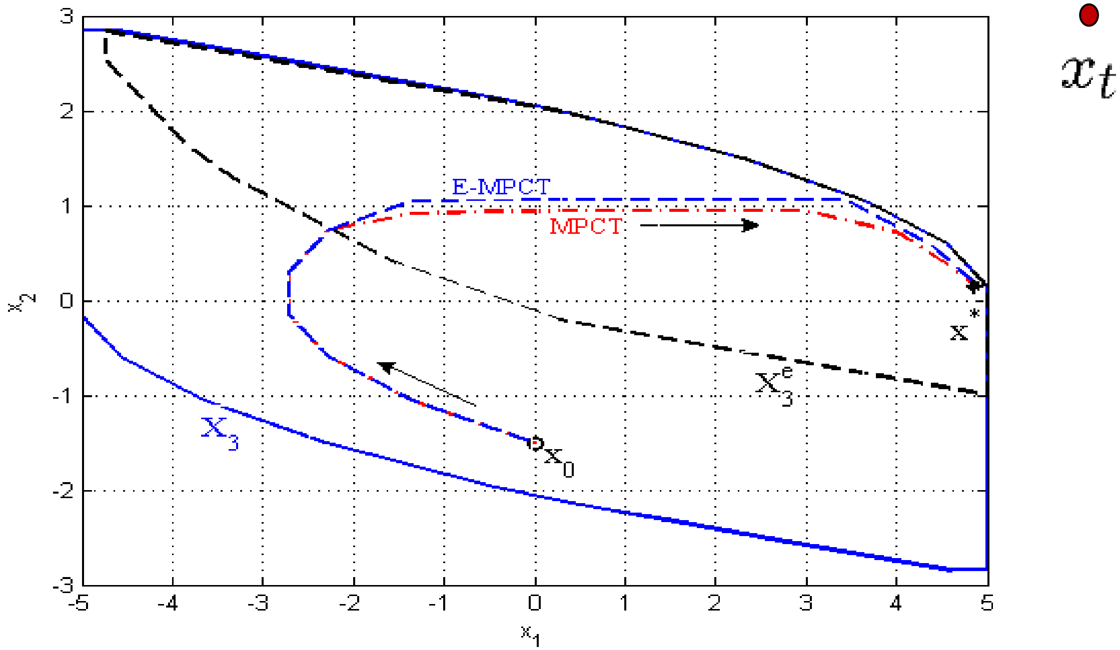
Measure	E-MPC	MPCT	E-MPCT
Φ	226.7878	304.3342	226.7878

$$\Phi = \sum_{k=0}^{T} |x(k) - x_t|_Q^2 + |u(k) - u_t|_R^2 - (|x^* - x_t|_Q^2 + |u^* - u_t|_R^2)$$

- **Feasibility:**

E-MPCT vs. MPCT

$N = 3$



Algunas referencias

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Appendix

Inherent Robustness

Inherent robustness

- In practice, there exists mismatches between the real plant and the model

$$x^+ = F(x, d) \qquad d \in D$$

Will the controlled real plant be stable?

- A primary requirement of the controller is that the nominal system

$$x(k+1) = F(x(k), 0)$$

is asymptotically stable (0-AS).

- A second objective :

$$\boxed{\text{Bounded uncertainty} \Rightarrow \text{Bounded evolution}}$$

$$\boxed{\text{Small uncertainty} \Rightarrow \text{Small effect}}$$

Input-to-State Stability

- Robust positively invariant (RPI) set :
  A set Γ is a RPI set if $F(x,d) \in \Gamma$ for all $x \in \Gamma$, $d \in D$.
  If $\Gamma \subseteq X$, the RPI set is called admissible.

- Input-to-state stability (Sontag'89)
  A system is ISS in the RPI Γ if there exists a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that

$$|x(j)| \leq \beta(|x(0)|, j) + \gamma(\|\mathbf{d}_{[j-1]}\|) \quad \forall x(0) \in \Gamma,\ \forall d(j) \in D$$

ISS for constrained systems

- ISS-Lyapunov function in Γ:
  Γ is a RPI set and there exists a function $V(x)$ such that
  - $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$, $\forall x \in \Gamma$
  - $V(f_\kappa(x,d)) - V(x) \leq -\alpha_3(|x|) + \lambda(|d|)$, $\forall x \in \Gamma$ and $d \in D$

  where $\alpha_1, \alpha_2,\ \alpha_3 \in \mathcal{K}_\infty,\ \lambda \in \mathcal{K}$

- Theorem:
  If the system admits a ISS-Lyapunov function in Γ, then it is ISS in Γ.

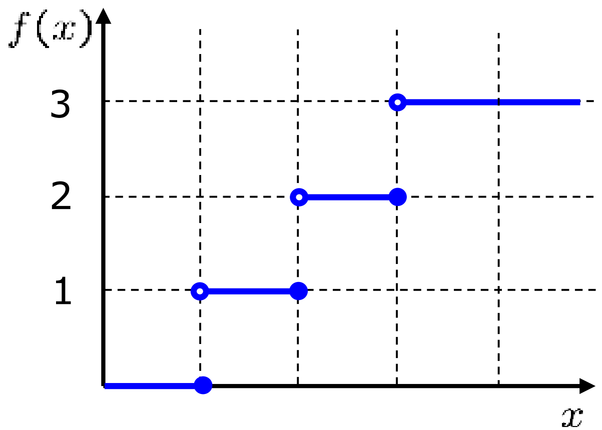
Inherent robustness

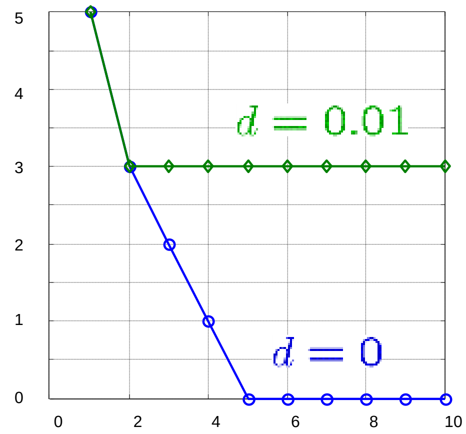
- ISS $\Rightarrow$ 0-$\mathcal{KLAS}$, but 0-$\mathcal{KLAS}$ $\not\Rightarrow$ ISS (discontinuity)

  nominal stability does not imply robustness

- Illustrative Example *(Kellet et al. '02)*

  Consider the dynamics: $x^+ = f(x) + d$



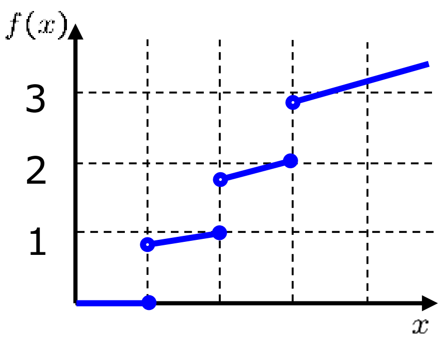


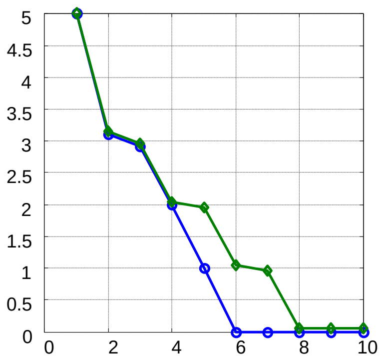
  - For any $x \in \mathbb{R}$ the nominal system is asymptotically stable
  - For any constant signal $d(k) > 0$, the state remains in a non-zero steady state.

The system is not ISS

Inherent robustness

- $0\text{-}\mathcal{KLAS} \Rightarrow$ ISS if one of the following holds:
  - The Lyapunov function of the nominal system is **uniformly continuous**.
  - $F(x,d)$ is **uniformly continuous** in x.





$V(x) = |x|$ is a unif. cont. Lyapunov function for all $x \in \mathbb{R}$

The system is ISS

Inherent robustness of MPC

- The MPC control law can be ensure $\mathcal{KLAS}$ of the nominal system
- But there may exist mismatches between the model and the real plant:
  (Variation on the parameters, External disturbances, unmodelled dynamics, etc.)
- The control law $\kappa(x)$ and/or $V_N^*(x)$ may be discontinuous, (even for continuous model functions) *[Meadows at al. 95]*

will the closed-loop real plant be robustly stable?

Illustrative example

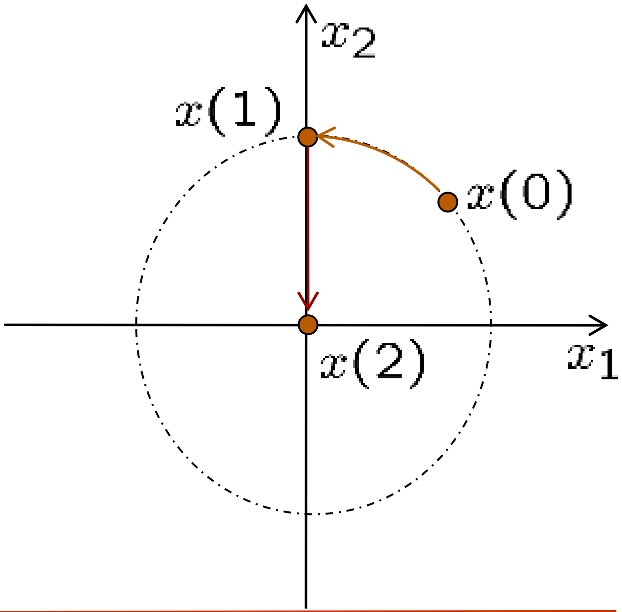
[Grimmet al.'04]

Consider the following constrained system

$$x^+ = \begin{bmatrix} x_1(1-u) \\ |x|u \end{bmatrix}, \quad u \in [0,1]$$

This is globally 0-AS by MPC with $L(x,u) = |x|^2 + |u|^2$, terminal equality constraint and N=2.

For all $x(0)$, the sequence $\mathbf{u} = \{1, 0\}$ is the only sequence such that $\phi(2, x, \mathbf{u}) = 0$

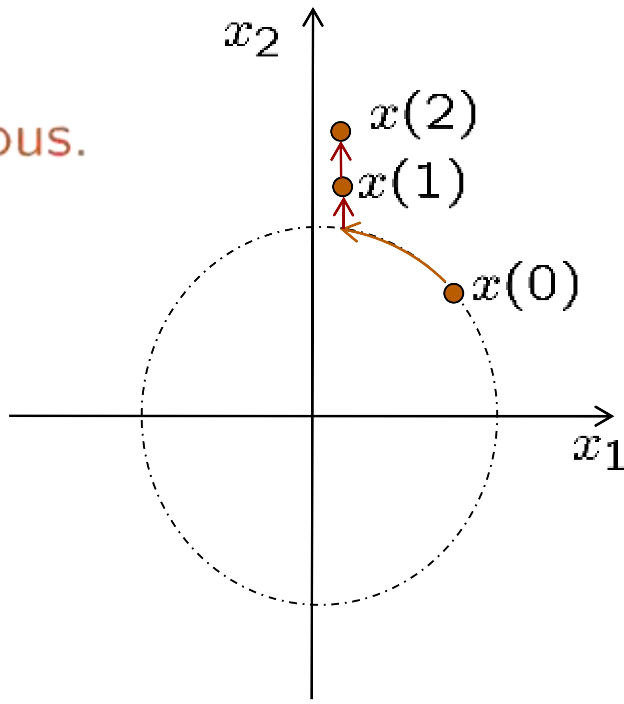


Then the MPC control law is

$$\kappa_2(x) = \begin{cases} 1 & x_1 \neq 0 \\ 0 & x_1 = 0 \end{cases}$$

The optimal cost function is:

$$V_N^0(x) = \begin{cases} 2|x|^2 + 1 & x_1 \neq 0 \\ |x|^2 & x_1 = 0 \end{cases}$$



Note that $V_N^0(x)$ and $\kappa_2(x)$ are discontinuous.

Robustness $: x^+ = f(x) + d$,
where $d_1(k) = d_2(k) = \epsilon$

$$\begin{aligned} x_1(k) &= \epsilon \\ x_2(k+1) &= |x(k)| + \epsilon \end{aligned}$$

and hence $x_2(k) \to \infty$.

The controlled system is not ISS $\Rightarrow$ **Zero robustness**

ISS of nominal MPC

- Theorem:
  - Assume that
    - $f(x,u,d)$ is uniformly continuous in d for all $(x,u) \in Z$, $d \in D$.
    - The stabilizing assumption holds
  - Then the closed-loop system is ISS in a closed set $\Omega_r \subset X_N$ (for small enough uncertainties) if one of the next conditions holds

a) Function $f(x, \kappa_N(x), 0)$ is unif. cont. in x for all $x \in X_N$.

b) The optimal cost $V_N^*(x)$ is unif. cont. in X_N.

ISS of nominal MPC

- Some conditions for uniform continuity of $V_N^*(x)$
  - $f(x,u,d)$ is unif. cont. in x
  - $L(x,u)$ and $V_f(x)$ are unif. cont. in x

a) $P_N(x)$ with a compact convex set of constraints.

- Linear systems and polytopic constraints

b) $P_N(x)$ unconstrained on the states:

- Constraint sets: $Z \triangleq \mathbb{R}^n \times U$ where $U \subseteq \mathbb{R}^m$ and $X_f \triangleq \mathbb{R}^n$

c) $P_N(x)$ with constrained on the states:

- There exists a level set Ω_r where the constraints on the states are not active